1. a) $F(-\infty) = a = 0$

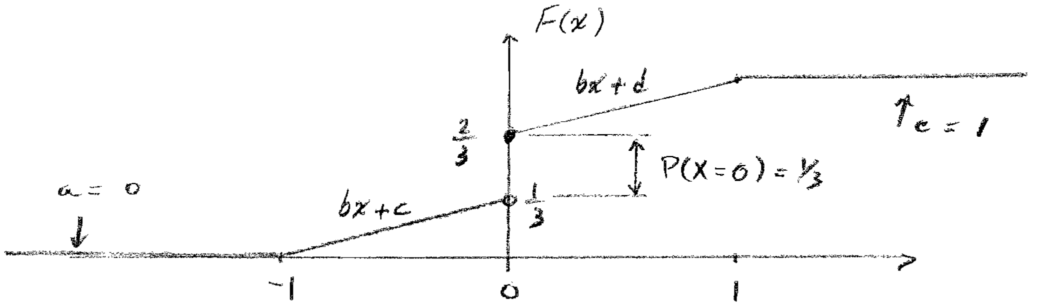
$F(+\infty) = e = 1$

$P(X=-1) = 0 \Longrightarrow F(x)$ cont. at $x=-1 \Longrightarrow -b+c=0$

$P(X=0) = 1/3 \Longrightarrow$ jump of $1/3$ at $x=0 \Longrightarrow d = c + 1/3$

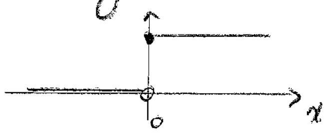
$P(X=1) = 0 \Longrightarrow F(x)$ cont. at $x=1 \Longrightarrow b+d=1$

The solution of these 3 equations is easily obtained as $b = c = 1/3$ and $d = 2/3$. The graph of $F(x)$ is shown below

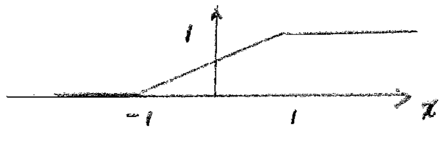


b) $F(x)$ is the linear superposition of a discrete CDF, say $F_d(x)$, and a continuous CDF, say $F_c(x)$:

$$F_d(x) = u(x)$$



$$F_c(x) = \begin{cases} 0, & x \le -1 \\ (x+1)/2, & -1 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$$



By inspection, we find

$$F(x) = \alpha F_d(x) + \beta F_c(x)$$
$$\alpha = 1/3 \quad \text{and} \quad \beta = 2/3$$

This shows that X is a mixed RV.

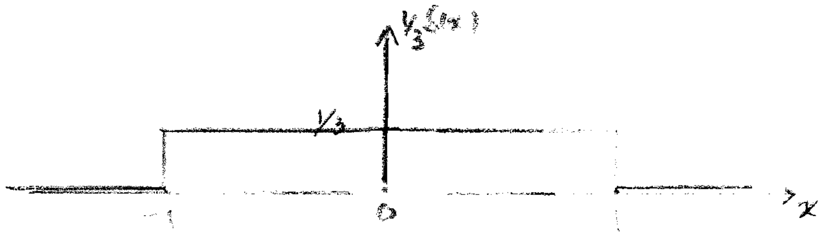
c) $$f(x) = \frac{d}{dx} F(x)$$

$$= \alpha f_d(x) + \beta f_c(x)$$

$$f_d(x) \triangleq \frac{d}{dx} F_d(x) = \delta(x) \quad \text{(unit impulse)}$$

$$f_c(x) = \frac{d}{dx} F_c(x) = \frac{1}{2}[u(x+1) - u(x-1)]$$

$$f(x) = \frac{1}{3}\delta(x) + \frac{1}{3}[u(x+1) - u(x-1)]$$



d) $$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = 0 \quad \text{(symmetry)}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{1}{3}\int_{-\infty}^{\infty} x^2 \delta(x)\,dx + \frac{1}{3}\int_{-1}^{1} x^2\,dx$$

$$= \frac{1}{3}\,\frac{x^3}{3}\Big|_{-1}^{1}$$

$$= \frac{2}{9}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2}{9}$$

Problem 2

(a) Since $P(T \geq 5°C) = 0.5$ we have $\boxed{E(T) = 5°C}$

$$P(T \geq 10°C) = P\Big(\underbrace{\frac{T - 5°C}{\sigma(°C)}}_{Z \sim N(0,1)} \geq \frac{10°C - 5°C}{\sigma(°C)}\Big) = 0.3085$$

$$\Rightarrow P\left(Z \geq \frac{5°C}{\sigma}\right) = 0.3085 \quad \text{(from table)}$$

$$\Rightarrow P\left(Z \leq \frac{5°C}{\sigma}\right) = 0.6915$$

$$\Rightarrow \frac{5°C}{\sigma} = 0.5 \Rightarrow \sigma = 10°C \quad \text{(standard deviation)}$$

The variance is $\boxed{\sigma^2 = 100\ (°C)^2}$

(b) $$P(-5°C < T < 0°C) = P\Big(\frac{-5°C - 5°C}{10°C} < \underbrace{\frac{T - 5°C}{10°C}}_{Z \sim N(0,1)} < \frac{0°C - 5°C}{10°C}\Big)$$

$$= P(-1 < Z < -0.5)$$

$$= \Phi(-0.5) - \Phi(-1) \quad (\Phi(\cdot)\text{: standard Gaussian CDF})$$

$$= 1 - \Phi(0.5) - [1 - \Phi(1)]$$

$$= \Phi(1) - \Phi(0.5) = 0.8413 - 0.6915$$

$$\Rightarrow P(-5°C < T < 0°C) = 0.1498$$

(c) $T_F \rightarrow$ RV representing temperature in °F. Since T_F is a linear function of T (the temperature in °C):

$$E(T_F) = \frac{9}{5}E(T) + 32 = \frac{9}{5} \times 5 + 32 = 41(°F)$$

$$Var(T_F) = \left(\frac{9}{5}\right)^2 Var(T) = 4 \times 81 = 324(°F)^2$$

Problem 3 (a) The pdf of X is $f_X(x) = \lambda e^{-\lambda x} U(x)$

$\rightarrow$ step function

Since $E(X) = 5\,yrs = \frac{1}{\lambda} \Rightarrow \lambda = 0.2\,yrs^{-1}$

$$P(X < 1\,yrs) = \underbrace{1 - e^{-0.2 \times 1}}_{\text{exponential CDF}} = 1 - 0.8187 = 0.1813$$

(b) Propability of a <u>single</u> spark plug working after a year: $p = 1 - P(X \leq 1\,yrs)$
$= 0.8187$

Let Z be the number of spark plugs (out of 20) that work after a year. Then, Z is Binomial $B(\overset{n}{20}, \overset{p}{0.8187})$

So, $E(Z) = n \cdot p = 20 \times 0.8187 = 16.374$ spark plugs

(c) Y is a discrete RV with set of possible values $R_Y = \{1, 2, 3, ...\}$
PMF of $Y \rightarrow p(y)$

$p(y) = 0$, $y \neq 1, 2, 3, ...$
For $y = 1, 2, 3, ...$

$$p(y) = P(Y = y) = P(y - 1 < X \leq y) = \int_{y-1}^{y} 0.2\, e^{-0.2x}\, dx$$

$$= \left[1 - e^{-0.2x}\right]_{y-1}^{y} = 1 - e^{-0.2y} - 1 + e^{-0.2(y-1)}$$

$$= e^{-0.2y} \cdot e^{0.2} - e^{-0.2y}$$

$$= e^{-0.2y}(e^{0.2} - 1)$$