McGILL UNIVERSITY
Faculty of Engineering

FINAL EXAMINATION
Fall 2006

ECSE 305, Section 001 (CRN 583)
PROBABILITY AND RANDOM SIGNALS I
DATE: Thursday, December 21, 2006
TIME: 9:00 – 12:00

Examiner: Prof. Benoît Champagne
Signature:

Associate Examiner: Prof. Yannis Psaromiligkos
Signature:

INSTRUCTIONS:

- This is a CLOSED BOOK examination.
- Faculty standard calculator permitted ONLY.
- This examination paper consists of 5 printed pages, including: a cover page, 6 questions and a Table in appendix. Ensure that you have a complete examination before starting.
- Answer ALL questions. Use one or more Answer Booklets for your solutions.
- You MUST RETURN this examination paper.

1. A box contains 4 light bulbs, two of which are defective. We want to find the defective bulbs by randomly testing them one at a time (without replacement). Let N_1 be the random variable denoting the number of tests until we find the first defective bulb, and let N_2 denote the number of tests until we find the second defective bulb. 16 marks

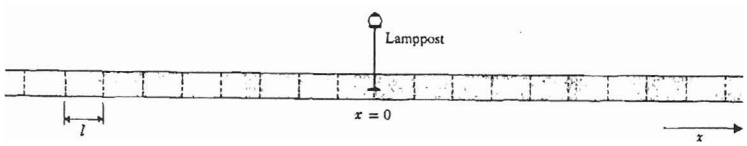
(a) Find the marginal probability mass function (PMF) of N_1, denoted $p_1(n_1)$.

(b) Find the conditional PMF of N_2 given N_1, denoted $p_{2|1}(n_2|n_1)$. Present your answers in tabular form.

(c) Find the joint PMF of N_1 and N_2. Again, present your answers in a table.

(d) Find the marginal PMF of N_2.

2. A drunk student starts out from a lamppost located near a sidewalk (see Figure below). Each step he takes is of equal length $l = 1$m. The student is however so drunk that the direction of each step, i.e. whether it is to the right (positive x) or to the left, is completely independent of the preceding steps. Assume that each time the student takes a step, the probability of a step to the right is p, while that of a step to the left is $q = 1 - p$. We are interested in the probability distribution of the final displacement of the student after he has taken n steps, where n is a given positive integer. 16 marks



(a) Out of n steps, let K denote the total number of steps to the right. Find the probability mass function of K and indicate the range of its possible values.

(b) Let X denote the final displacement (in units of l) of the student after taking n steps. Obtain a relationship between X and K, and find the expected value and standard deviation of X.

(c) In the special $N = 3$ and $p = \frac{1}{2}$, find the PMF of X and clearly indicate the range of its possible values.

3. Let X be a random variable uniformly distributed on $[0,1]$. Also, let Y be a random variable exponentially distributed with parameter $\lambda = 1$, independent of X. The random variables W and Z are defined as follows: 18 marks

$$\begin{aligned} W &= X^2 \\ Z &= XY \end{aligned}$$

(a) Find the joint PDF, say $f(x,y)$, of X and Y.

(b) Using the method of transformations find the joint probability density function (PDF) of W and Z, say $g(w,z)$.

(c) Sketch the region E of the (w,z)-plane over which the PDF $g(w,z)$ is non-zero.

(d) Find the marginal PDF of W, say $h(w)$.

4. The random variables X and Y are are uniformly distributed over the region D given by 16 marks

$$D = \{(x,y) \in [0,1]^2 : y \leq 1 - x\} \cup \{(x,y) \in [-1,0]^2 : y \geq -(1+x)\}$$

(a) Sketch the region D.

(b) Find the joint PDF $f(x,y)$ of X and Y.

(c) Find and sketch the marginal PDF of Y. What is the mean value of Y?

(d) Find the correlation coefficient of X and Y.

5. The customer service time at a certain bank can be modeled as a RV with PDF 18 marks

$$f(x) = \begin{cases} \frac{1}{\alpha} e^{-x/\alpha}, & x > 0 \\ 0, & x \leq \tau \end{cases} \tag{1}$$

where $\alpha > 0$ is an unknown parameter.

In order to estimate α, you decide to poll n customers who recently visited the bank and apply a so-called *maximum log-likelihood estimator* on the data. Let X_i denote service time of the ith customer ($i = 1, \ldots, n$) in the poll. Assume that the RVs X_i are independent and identically distributed as in equation (1).

(a) What are the expected value and variance of the service time X in (1)?

(b) Find $f(x_1, \ldots, x_n)$, the joint PDF of the RVs $X_1, \ldots, X_n$.

(c) Assuming that all the variables x_i are positive, find an expression for the log-likelihood function, defined as

$$L(x_1, \ldots, x_n) \triangleq \ln f(x_1, \ldots, x_n)$$

(d) The value of α at which $L(x_1, \ldots, x_n)$ attains a maximum, denoted $\hat{\alpha}$, is called maximum likelihood estimator of α. Obtain $\hat{\alpha} = \hat{\alpha}(x_1, \ldots, x_n)$ as a function of $x_1, \ldots, x_n$.

(e) Define the RV

$$Y = \hat{\alpha}(X_1, \ldots, X_n)$$

Find an approximation for the PDF of Y, say $g(y)$, valid in the limit of n large.

(f) How large should be n so that Y is within 5% of its mean value at least 19 times out of 20.

6. A white noise process $X(t)$ with power spectral density (PSD) level P_W is passed through an LTI system with frequency response $H(\omega)$, such that the resulting output signal $Y(t)$ has the following PSD: **16 marks**

$$S_y(\omega) = \begin{cases} P_Y, & |\omega| < \Omega/2 \\ 0, & |\omega| > \Omega/2 \end{cases}$$

where $\Omega > 0$ is the desired output bandwidth.

(a) Assuming that the phase response of the LTI filter is zero, i.e. $\measuredangle H(\omega) = 0$, find its impulse response $h(t)$.

(b) Find the autocorrelation function of the process $Y(t)$.

(c) $Y(t)$ is passed through a modulator whose output is given by

$$Z(t) = Y(t)\cos(\omega_o t + \Theta)$$

where ω_o is a deterministic carrier frequency and Θ is a random phase independent of $Y(t)$, uniformly distributed within $(-\pi, \pi]$. Show that $Z(t)$ is WSS. (Hint: $2\cos(A)\cos(B) = \cos(A - B) + \cos(A + B)$).

Appendix: Table of values of the standard normal CDF

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990