McGILL UNIVERSITY
Faculty of Engineering

FINAL EXAMINATION
Fall 2005

ECSE 305, Section 001 (CRN 583)
PROBABILITY AND RANDOM SIGNALS I
DATE: Thursday, December 8, 2005
TIME: 9:00 – 12:00

Examiner: Prof. Benoît Champagne

Signature:

Associate Examiner: Prof. Yannis Psaromiligkos

Signature:

INSTRUCTIONS:

- This is a CLOSED BOOK examination.
- Faculty standard calculator permitted ONLY.
- This examination paper consists of 5 printed pages, including: a cover page, 6 questions and an appendix. Ensure that you have a complete examination before starting.
- Answer ALL questions. Use one or more Answer Booklets for your solutions.
- You MUST RETURN this examination paper.

1. A bank has two cashiers A and B. The time it takes cashier A to serve a customer is exponentially distributed with parameter $\lambda_A = \frac{1}{8}\text{minutes}^{-1}$, while the time it takes cashier B to serve a customer is again exponentially distributed but with parameter $\lambda_B = \frac{1}{5}\text{minutes}^{-1}$. Since cashier A is closer to the bank's entrance it has been observed that 70% of the bank's customers go to him. $\boxed{\text{20 marks}}$

(a) Let X denote the service time of a randomly selected customer. Find the cumulative distribution function (CDF) of X.

(b) Find the probability density function (PDF) of X.

(c) What is the mean of X?

(d) A customer just came out of the bank and tells you that in his case the service time was less than 6 minutes. What is the probability that he was served by cashier A?

(e) If the same customer had asked you to guess if he was served by cashier A or B, what would be your guess? Justify your answer.

2. A random experiment consists of two sequential steps as follows: (1) a fair die is tossed until 6 shows up for the first time; we let X denote the number of tosses; (2) the die is then tossed again X times; we let Y denote the total number of 6 observed in this second step. $\boxed{\text{20 marks}}$

(a) Find the marginal PMF of RV X. What kind of discrete RV is X?

(b) Find the joint PMF of RVs X and Y, say $p(x, y)$.

(c) Find the probability that $Y = 0$.

(d) Let RV $Z = Y/X$. Find the expected value of Z.

3. Random variables X and Y are jointly uniform over the region $\boxed{\text{20 marks}}$

$$D = \{(x, y) \in \mathbb{R}^2 : |x| < 1 \text{ and } x - \epsilon < y < x + \epsilon\}$$

where ϵ is a small positive number ($0 < \epsilon < 1$).

(a) Sketch the region D.

(b) Find the joint PDF of X and Y as well as the marginal PDF of X.

(c) Find the covariance of X and Y.

(d) Are X and Y independent?

(e) It can be shown that $E(Y^2) = \frac{1}{3}(1+\epsilon^2)$. Using this result, find the correlation coefficient of X and Y, i.e. $\rho(X,Y)$.

(f) Find the limit of $\rho(X,Y)$ as $\epsilon \to 0$ and provide an intuitive justification for your result.

4. The random variables X and Y are jointly uniform with the following PDF: $\boxed{\text{20 marks}}$

$$f(x,y) = \begin{cases} 0.25, & -1 < x < 1 \text{ and } -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The random variables W and Z are defined as follows:

$$\begin{aligned} W &= X^2 Y \\ Z &= Y \end{aligned}$$

(a) Using the method of transformations find the joint PDF of W and Z, say $g(w,z)$.

(b) Sketch the region E of the (w,z)-plane over which the PDF $g(w,z)$ is non-zero.

5. Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed RVs with common marginal PDF $\boxed{\text{20 marks}}$

$$f(x) = \begin{cases} 1/2, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Define RV $Y = X_1 + X_2 + \cdots + X_n$ and let $g(y,n)$ denote the PDF of Y for a given integer value of n.

(a) Sketch the graph of $g(y,n)$ for $n = 1, 2$ and 4 (no lengthy calculations required).

(b) For n large, give a suitable approximation to $g(y,n)$.

(c) In the case $n = 12$, find the minimum value of a scaling factor $c > 0$ such that $P(|Y/c| \leq 1) \leq 0.95$.

6. Consider the following random process: $\boxed{\text{20 marks}}$

$$X(t) = A\cos(\omega_1 t + \Theta) + B\cos(\omega_2 t + \Theta)$$

where ω_1, ω_2 are deterministic angular frequencies with $\omega_1 < \omega_2$. The random variables A and B are such that $E\{A\} = E\{B\} = 0$, $Var\{A\} = Var\{B\} = 1$ and $E\{AB\} = \rho$. The random phase Θ is uniformly distributed on $[0, \pi]$ and is independent of A and B.

(a) Find the mean function $\mu_X(t)$ of $X(t)$.

(b) Find the autocorrelation function $R_X(t, u)$ of $X(t)$. *Hint*: $\cos(a)\cos(b) = \frac{1}{2}\cos(a - b) + \frac{1}{2}\cos(a + b)$.

(c) For what value of ρ is $X(t)$ a wide-sense stationary process?

(d) For the value of ρ you found in (c), find and sketch the power spectral density $S_X(\omega)$ of $X(t)$.

(e) Suppose that the WSS process $X(t)$ is passed through an ideal lowpass filter with magnitude frequency response

$$|H(\omega)| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

where $\omega_c = \frac{1}{2}(\omega_1 + \omega_2)$. Find and sketch the power spectral density of the output process $Y(t)$.

Appendix: Table of values of the standard normal CDF

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990