ECSE 305A: Probability and Random Signals I
Problem Set 9
solutions

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November 29, 2006

1. (a)

$$\begin{aligned}
F(x,y) &= \int_0^1 \int_0^1 f(x,y)dxdy \\
&= \int_0^1 \int_0^1 cx^k y^l dxdy \\
&= \int_0^1 c \left.\frac{x^{k+1}}{k+1}\right]_0^1 y^l dy \\
&= \frac{c}{k+1} \left.\frac{y^{l+1}}{l+1}\right]_0^1 \\
&= \frac{c}{(k+1)(l+1)} \\
&= 1. \\
&\Rightarrow c = (k+1)(l+1).
\end{aligned}$$

(b)

$$\begin{aligned}
P(Y > X) &= c\int_0^1 \int_x^1 x^k y^l dxdy \\
&= c_0^1 \left.\frac{y^{l+1}}{l+1}\right]_x^1 x^k dx \\
&= c\int_0^1 \left(\frac{1}{l+1} - \frac{x^{l+1}}{l+1}\right) x^k dx \\
&= (k+1)(l+1)\left[\frac{1}{(k+1)(l+1)} - \frac{1}{(l+1)(k+l+2)}\right]_0^1 \\
&= 1 - \frac{k+1}{k+l+2}
\end{aligned}$$

If $l > k$ then $P(Y > X) > 1/2$.

(c)

$$f_Y(y) = \int_0^1 cx^k y^l dx = c \left.\frac{x^{k+1}}{k+1}\right]_0^1 = (l+1)y^l;$$

$$f_X(x) = \int_0^1 cx^k y^l dy = c \left.\frac{y^{l+1}}{l+1}\right]_0^1 = (k+1)x^k.$$

2. (a)

$$\begin{aligned} \textstyle\sum_{i\in R_I}\sum_{j\in R_J} p(i,j) &= 1 \\ (0+1+1+1+2+2+1+2+2)c &= 1 \\ &\Rightarrow c=\tfrac{1}{12} \end{aligned}$$

(b)

$$P_{I|J}(i,j) = \frac{P(I=i,J=j)}{P(J=j)} = \frac{\frac{i^2+j^2}{12}}{P(J=j)}$$

$$P_J(j) = \begin{cases} \frac{1+2+2}{12} = \frac{5}{12}, & j=-1 \\ \frac{0+1+1}{12} = \frac{2}{12}, & j=0 \\ \frac{1+2+2}{12} = \frac{5}{12}, & j=1 \end{cases}$$

$$P_{I|J}(i,j) = \begin{cases} \frac{i^2+1}{5}, & j=-1 \\ \frac{i^2}{2}, & j=0 \\ \frac{i^2+1}{5}, & j=1 \end{cases}$$

(c)

$$P(|I|=1|J=0) = P(I=1|J=0)+P(I=-1|J=0) = 1/2+1/2 = 1$$

3. See Fig. 1.

4.

$$\begin{aligned} F_Y(y) &= \textstyle\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_X}{\sigma_x})^2-2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y})+(\frac{y-\mu_Y}{\sigma_Y})^2]} dx \\ &= \textstyle\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_X}{\sigma_x})^2-2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y})+\rho^2(\frac{y-\mu_Y}{\sigma_Y})^2-\rho^2(\frac{y-\mu_Y}{\sigma_Y})^2+(\frac{y-\mu_Y}{\sigma_Y})^2]} dx \\ &= \textstyle\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[((\frac{x-\mu_X}{\sigma_x})-\rho(\frac{y-\mu_Y}{\sigma_Y}))^2+(1-\rho^2)(\frac{y-\mu_Y}{\sigma_Y})^2]} dx \\ &= \frac{e^{-\frac{1}{2}(\frac{y-\mu_Y}{\sigma_Y})^2}}{2\pi\sigma_Y} \textstyle\int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}((\frac{x-\mu_X}{\sigma_x})-\rho(\frac{y-\mu_Y}{\sigma_Y}))^2} dx \\ &= \frac{e^{-\frac{1}{2}(\frac{y-\mu_Y}{\sigma_Y})^2}}{\sqrt{2\pi}\sigma_Y} \end{aligned}$$

5.

$$P(X \le 0|y=1) = \Phi(\frac{0-1}{1}) = \Phi(-1) = 1-\Phi(1) = 0.1587$$

Experimental value : 0.1570

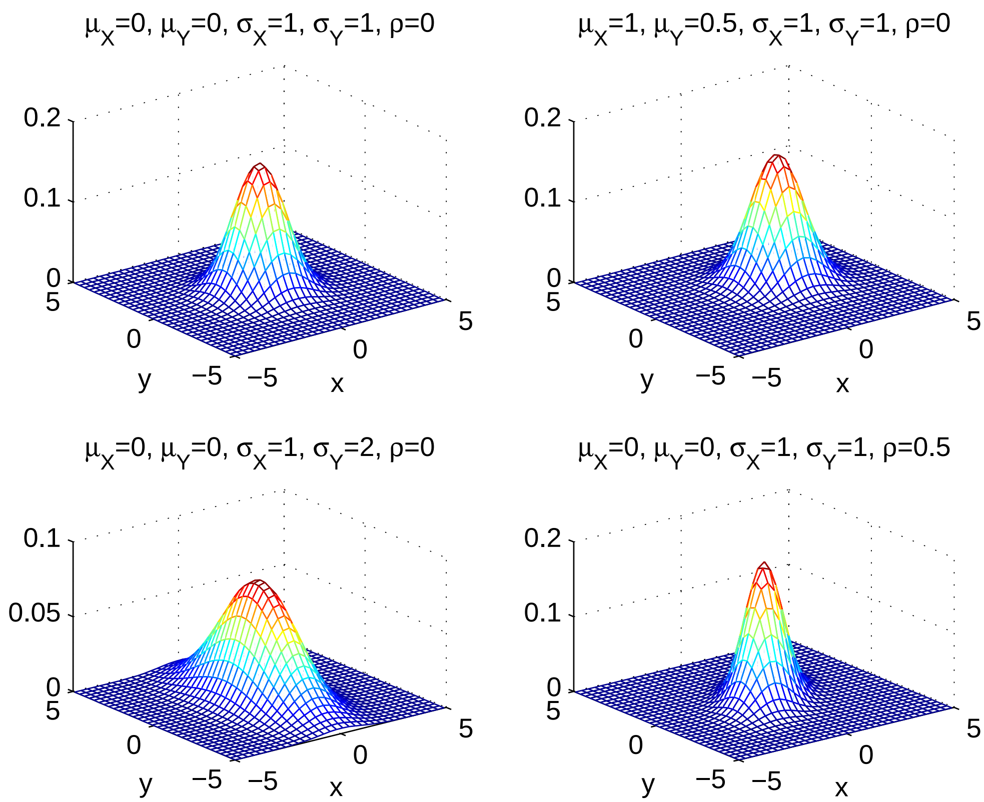

Figure 1: Question 3

6. RV X and Y are dependent. This can be seen by following example: Consider $P(X = 8, Y = 8)$ which is equal to zero but $P(X = 8)P(Y = 8) = \frac{\binom{13}{8}}{\binom{52}{8}} \frac{\binom{13}{8}}{\binom{52}{8}} \neq P(X = 8, Y = 8)$.

7. (a)

$$f(x, y) = \frac{1}{(b-a)(d-c)}$$

See Fig. 2.

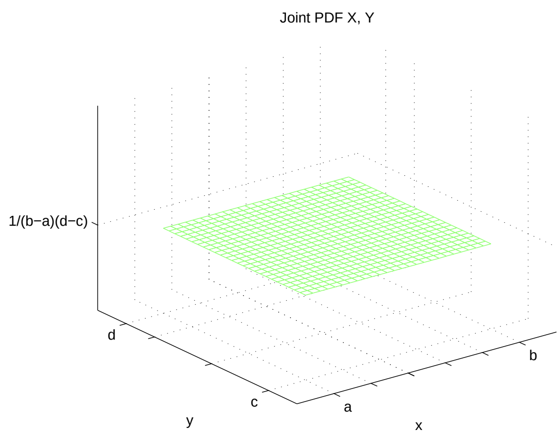

Figure 2: Question 7 (a)

(b)

$$F(x, y) = \int_a^x dt \int_c^y du f(t, u) = \frac{(y-c)(x-a)}{(b-a)(d-c)}$$

See Fig 3.

(c)

$$f_Y(y) = \int_a^b \frac{1}{(b-a)(c-d)} dx = \frac{(b-a)}{(b-a)(c-d)} = \frac{1}{d-c}$$

$$f_X(x) = \int_c^d \frac{1}{(b-a)(c-d)} dy = \frac{(d-c)}{(b-a)(c-d)} = \frac{1}{b-a}$$

Therefore, $f(x, y) = f_X(x) f_Y(y)$ and two RV are independent.

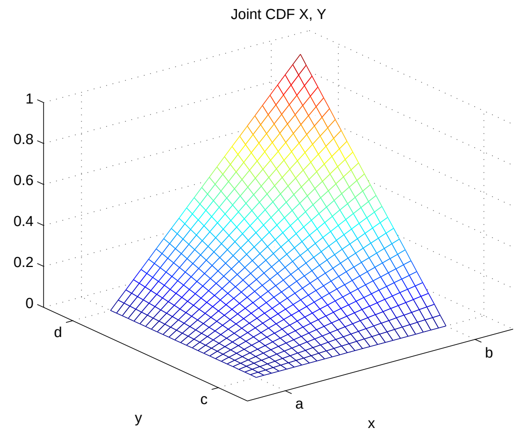

Figure 3: Question 7 (b)

8. Let $f(x, y)$ be the joint probability density function of X and Y. Clearly,

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{\frac{-x^2}{2\sigma^2}} e^{\frac{-y^2}{2\sigma^2}}$$

Since, $x = r\cos\theta$ and $y = r\sin\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0$$

$$g(r,\theta) = f(r\cos\theta, r\sin\theta)|r| = \frac{1}{2\pi\sigma^2} r e^{\frac{-r^2}{2\sigma^2}}$$

Now

$$g_R(r) = \int_0^{2\pi} \frac{1}{2\pi\sigma^2} r e^{-\frac{r^2}{2\sigma^2}} d\theta = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, \quad r > 0$$

and

$$g_\Theta(\theta) = \int_0^{\infty} \frac{1}{2\pi\sigma^2} r e^{-\frac{r^2}{2\sigma^2}} dr = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi$$

Therefore, $g(r,\theta) = g_R(r) g_\Theta(\theta)$, showing that R and Θ are independent random variables

9.

$$Z = \max(X,Y) = \begin{cases} X & if X > Y \\ Y & if X \leq Y \end{cases}$$

$$
\begin{aligned}
F_Z(\alpha) &= P(z \leq \alpha) \\
&= P(\max(X,Y) \leq \alpha) \\
&= P(X \leq \alpha, Y \leq \alpha) \\
&= F_{X,Y}(\alpha,\alpha) \\
&= F_x(\alpha)F_Y(\alpha) \\
&= (1-e^{-\lambda\alpha})^2
\end{aligned}
$$