ECSE 305A: Probability and Random Signals I
Problem Set 8
solutions

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1. The MacLaurin's series for $M_X(t)$ is given by

$$\begin{aligned}
M_X(t) &= \textstyle\sum_{n=0}^{\infty} \frac{M_X^{(n)}(0)}{n!} t^n \\
&= \textstyle\sum_{n=0}^{\infty} \frac{E(X^n)}{n!} t^n \\
&= \textstyle\sum_{n=0}^{\infty} (n+1)(2t)^n \\
&= \frac{1}{2} \textstyle\sum_{n=0}^{\infty} \frac{d}{dt}(2t)^{n+1} \\
&= \frac{1}{2}\frac{d}{dt}[\textstyle\sum_{n=0}^{\infty}(2t)^{n+1}] \\
&= \frac{1}{2}\frac{d}{dt}[\textstyle\sum_{n=-1}^{\infty}(2t)^{n+1} - 1] \\
&= \frac{1}{2}\frac{d}{dt}[\textstyle\sum_{n=0}^{\infty}(2t)^{n} - 1] \\
&= \frac{1}{2}\frac{d}{dt}[\frac{1}{1-2t} - 1] \\
&= \frac{1}{(1-2t)^2} \\
&= (\frac{1/2}{1/2-t})^2.
\end{aligned}$$

We see that for $t < 1/2$, $M_X(t)$ exists; furthermore, it is the moment-generating function of a gamma random variable with parameters $r = 2$ and $\lambda = 1/2$.

2. (a)

$$\begin{aligned}
\psi(\omega) &= \textstyle\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx \\
&= \textstyle\int_0^1 6x(1-x)dx \\
&= \frac{e^{-j\omega x}}{-j\omega}\left(x - \frac{1}{-j\omega}\right) - \frac{e^{-j\omega x}}{-j\omega}\left(x^2 - \frac{2x}{-j\omega} + \frac{2}{-\omega^2}\right)\Big]_0^1 \\
&= \frac{12je^{-j\omega}}{w^3} - \frac{6e^{-j\omega}}{w^2} - \frac{6}{\omega^2} - \frac{12j}{w^3}
\end{aligned}$$

(b)

$$\mu = j\psi'(0)$$

Substituting the Taylor series for $e^{j\omega} = 1 - j\omega - \frac{\omega^2}{2} + \frac{j\omega^3}{6} + \frac{\omega^4}{24} - \frac{j\omega^5}{30} + \cdots$, we have

$$\begin{aligned}
\psi(\omega) &= \tfrac{12j}{w^3}(1 - j\omega - \tfrac{\omega^2}{2} + \tfrac{j\omega^3}{6} + \tfrac{\omega^4}{24} - \tfrac{j\omega^5}{30} + \cdots) \\
&\quad -\tfrac{6}{w^2}(1 - j\omega - \tfrac{\omega^2}{2} + \tfrac{j\omega^3}{6} + \tfrac{\omega^4}{24} - \tfrac{j\omega^5}{30} + \cdots) - \tfrac{6}{\omega^2} - \tfrac{12j}{w^3} \\
&= \tfrac{12j}{\omega^3} + \tfrac{12}{\omega^2} - \tfrac{6j}{\omega} - 2 + \tfrac{j\omega}{2} + \tfrac{2\omega^2}{5} + \cdots \\
&\quad -\tfrac{6}{\omega^2} + \tfrac{6j}{\omega} + 3 - j\omega - 4\omega^2 + \cdots - \tfrac{6}{\omega^2} - \tfrac{12j}{\omega^3} \\
&= +1 - \tfrac{j\omega}{2} + \tfrac{22\omega^2}{5} + \cdots
\end{aligned}$$

$$\begin{aligned}
\mu &= j\psi'(0) \\
&= j\left(-\tfrac{j}{2} + \tfrac{44\omega}{5} + \cdots\right)\Big|_{\omega=0} \\
&= 1/2.
\end{aligned}$$

3. By definition,

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \textstyle\sum_{x=0}^{\infty} e^{tX} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \textstyle\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
&= e^{-\lambda} \exp(\lambda e^t) \\
&= \exp[\lambda(e^t - 1)].
\end{aligned}$$

From

$$M'_X(t) = \lambda e^t \exp[\lambda(e^t - 1)]$$

and

$$M''_X(t) = (\lambda e^t)^2 \exp[\lambda(e^t - 1)] + \lambda e^t \exp[\lambda(e^t - 1)],$$

we obtain $E(X) = M'_X(0) = \lambda$ and $E(X^2) = M''_X(0) = \lambda^2 + \lambda$. Therefore,

$$Var(X) = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

4. We have that

$$\begin{aligned}
M_{2X+1}(t) &= E[e^{(2X+1)t}] \\
&= e^t E(e^{2tX}) \\
&= e^t M_X(2t) \\
&= \tfrac{e^t}{1-2t}, \qquad t < \tfrac{1}{2}.
\end{aligned}$$

5. (a)

$$\begin{aligned}
\textstyle\sum_{x \in R_X} \sum_{y \in R_Y} p(x, y) &= 1 \\
(2 + 3 + 4 + 3 + 4 + 5 + 4 + 5 + 6)k &= 1 \\
&\Rightarrow k = \tfrac{1}{36}
\end{aligned}$$

(b)

$$\begin{aligned} P(X=1,Y<3) &= p(x=1,y=1)+p(x=1,y=2) \\ &= 2/36+3/36=5/36. \end{aligned}$$

$$\begin{aligned} P(X=1,Y\le 3) &= p(x=1,y=1)+p(x=1,y=2)+p(x=1,y=3) \\ &= 2/36+3/36+4/36=1/4. \end{aligned}$$

$$\begin{aligned} P(X=2) &= p(x=2,y=1)+p(x=2,y=2)+p(x=2,y=3) \\ &= 2/36+3/36+4/36=1/3. \end{aligned}$$

$$\begin{aligned} P(X<Y) &= p(x=1,y=2)+p(x=1,y=3)+p(x=2,y=3) \\ &= 3/36+4/36+5/36=1/3. \end{aligned}$$

$$\begin{aligned} P(X\le Y) &= P(X<Y)+p(x=1,y=1)+p(x=2,y=2)+p(x=3,y=3) \\ &= 1/3+4/36+5/36+3/36=2/3. \end{aligned}$$

6. (a)

$$\begin{aligned} \textstyle\sum_{x\in R_X}\sum_{y\in R_Y} p(x,y) &= 1 \\ ((1+1)+(1+9)+(4+9))k &= 1 \\ &\Rightarrow k=\tfrac{1}{25} \end{aligned}$$

(b)

$$p_X(x)=\begin{cases} \sum_{y\in R_Y} p(x=1,y)=2/25+10/25=12/25, & 1 \\ \sum_{y\in R_Y} p(x=2,y)=13/25, & 2 \end{cases}$$

$$p_Y(y)=\begin{cases} \sum_{x\in R_X} p(x,y=1)=2/25, & 1 \\ \sum_{x\in R_X} p(x,y=3)=23/25, & 3 \end{cases}$$

7. (a)

$$f_X(x)=\int_0^x 2dy=2x, \quad 0\le x\le 1$$

$$f_Y(y)=\int_y^1 2dx=2(1-y), \quad 0\le y\le 1$$

(b)

$$P(X<\frac{1}{2})=\int_0^{\frac{1}{2}} f_X(x)dx=\frac{1}{4}$$

$$P(X<2Y)=\int_0^1\int_{x/2}^x 2dydx=\frac{1}{2}$$

$$P(X=Y)=0$$

8. The problem is equivalent to the following: Two random numbers X and Y are selected at random and independently from $(0, l)$. What is the probability that $|X - Y| < X$? Let $S = \{(x, y) : 0 < x < l, 0 < y < l\}$ and

$$R = \{(x, y) \in S : |x - y| < x\} = \{(x, y) \in S : y < 2x\}$$

The desired probability is the area of R which is $3l^2/4$ divided by l^2. So the answer is $3/4$.

9.

$$\begin{aligned} P(Y \le X \text{ and } X^2 + Y^2 \le 1) &= \int_0^{\sqrt{2}} \int_0^x dxdy + \int_{\sqrt{2}}^1 \int_0^{\sqrt{1-x^2}} dxdy \\ &= \int_0^{\sqrt{2}} xdx + \int_{\sqrt{2}}^1 \sqrt{1 - x^2}dx \\ &= \pi/8. \end{aligned}$$

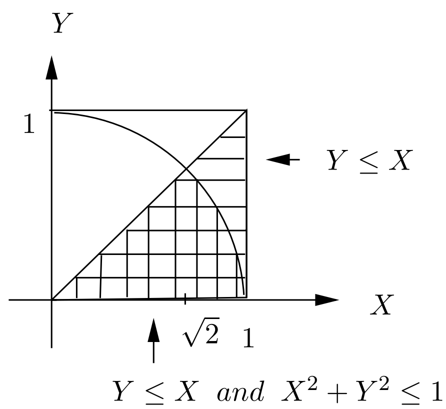

Figure 1: Question 9