ECSE 305A: Probability and Random Signals I
Problem Set 7
solutions

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November 8, 2006

1. If $I = \int_{-\infty}^{\infty} e^{-x^2/2} dx$, then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx\ dy. \end{aligned}$$

   Therefore,we can change the variables to polar coordinates, where $x = r\cos\theta$ and $y = r\sin\theta$ and $dx\ dy = r\ dr\ d\theta$. We have

$$\begin{aligned} I^2 &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} r\ dr\ d\theta &&= \int_0^{\infty} e^{-r^2/2}\left(\int_0^{2\pi} d\theta\right) dr \\ &= 2\pi \int_0^{\infty} r e^{-r^2/2}\ dr &&= 2\pi\left[-e^{-r^2/2}\right]_0^{\infty} \\ &= 2\pi. \end{aligned}$$

   Thus, $I = \sqrt{2\pi}$ and $\int_{-\infty}^{\infty} \phi(x)\ dx = 1$.

2. See Figure 1.

3. Let g be the density function of $|X - \mu|$. $g(t) = 0$ if $t < 0$; for $t \geq 0$,

$$g(t) = \sum_i f(x_i)|\frac{dx_i}{dt}| = 2 \times \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-t^2}{2\sigma^2}\right)$$

   Therefore,$G(t) = 0$ if $t < 0$ and for $t \geq 0$

$$G(t) = 2\ \Phi(\frac{t}{\sigma}) - 1.$$

   Hence,

$$E(|X - \mu|) = \int_0^{\infty} t g(t) dt.$$

substituting $u = t/\sigma$, we obtain

$$E(|X-\mu|) = \int_0^\infty \frac{2\sigma}{\sqrt{2\pi}} u e^{-u^2/2} du = \sigma\sqrt{\frac{2}{\pi}}.$$

4. Assuming that the chest sizes has normal distribution, the mean is 40 and the σ can be found, using the inflection points information,

$$\frac{42.5-40}{\sigma} = 1 \Rightarrow \sigma = 2.5.$$

If p is the probability that a random soldier has a chest more than 40 inches, then

$$p = P(X \geq 40) = P(\frac{X-40}{2.5} \geq \frac{40-40}{2.5}) = P(\frac{X-40}{2.5} \geq 0) = 0.5.$$

Therefore, the probability that form 50 randomly selected soldiers, 10 had a chest size of a least 40 is:

$$\binom{50}{10}(0.5)^{10}(0.5)^{40}.$$

5. (a)We want to find x so that $P(X \geq x) = 0.10$ or $P(X < x) = 0.90$.

$$P(\frac{X-75}{10} < \frac{x-75}{10}) = 0.90$$

Hence,

$$\Phi(\frac{x-75}{10}) = 0.90$$

From the table for $\Phi(x)$, we have that $\Phi(1.28) \approx 0.8997$. Thus, $x = 10 * 1.28 + 75 = 87.8$, which means he/she must score at least 87.8.
(b)

$$P(\frac{X-75}{10} < \frac{x-75}{10}) = 0.95 \text{ and } \Phi(1.59) \approx 0.9441 \Rightarrow x = 90.9.$$

$$P(\frac{X-75}{10} < \frac{x-75}{10}) = 0.98 \text{ and } \Phi(2.06) \approx 0.9803 \Rightarrow x = 95.6.$$

6. Assuming $Z = X - \mu$, then $Z \sim N(0, \sigma^2)$ and we are looking for $E[Z^n]$. The odd moments of Z are 0 because $f(-z) = f(z)$. Differentiating k times the identity

$$\int_{-\infty}^{\infty} e^{-\alpha z^2} = \sqrt{\frac{\pi}{\alpha}}$$

This yields

$$\int_{-\infty}^{\infty} z^{2k} e^{-\alpha z^2} = \frac{1 \times 3 \times \cdots \times (2k-1)}{2^k} \sqrt{\frac{\pi}{\alpha^{2k+1}}}$$

and with $\alpha = 1/2\sigma^2$. Therefore,

$$E(Z^n) = \begin{cases} 0 & n = 2k+1 \\ 1 \times 3 \times \cdots \times (n-1)\sigma^n & n = 2k \end{cases} \qquad k = 1, 2, \ldots$$

7.

$$P(\lfloor X \rfloor = n) = P(n \leq X < n+1) = \int_n^{n+1} \lambda e^{-\lambda x} dx = (e^{-\lambda})^n (1 - e^{-\lambda}).$$

This is the probability function of a geometric random variable with parameter $p = 1 - e^{-\lambda}$.

8.

$$\begin{aligned} P(|X - E(X)| \geq 2\sigma_X) &= P(|X - \tfrac{1}{\lambda}| \geq \tfrac{2}{\lambda}) \\ &= P(X - \tfrac{1}{\lambda} \geq \tfrac{2}{\lambda}) + P(X - \tfrac{1}{\lambda} \leq -\tfrac{2}{\lambda}) \\ &= P(X \geq \tfrac{3}{\lambda}) + P(X \leq -\tfrac{1}{\lambda}) \\ &= e^{-\lambda(3/\lambda)} + 0 = e^{-3} = 0.049787 \end{aligned}$$

9. If F denotes the probability distribution function of Y. If $t \leq 0$, then $F(t) = 0$. For $t > 0$,

$$\begin{aligned} F(t) &= P(Y \leq t) = P(X^2 \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) \\ &= \Phi(\sqrt{t}) - \Phi(-\sqrt{t}) = \Phi(\sqrt{t}) - \left[1 - \Phi(\sqrt{t})\right] = 2\Phi(\sqrt{t}) - 1. \end{aligned}$$

If f denotes the PDF of Y. For $t \leq 0$, $f(t) = 0$. For $t > 0$,

$$f(t) = F'(t) = 2 \cdot \frac{1}{2\sqrt{2}} \Phi'(\sqrt{t}) = \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-t/2} = \frac{(t)^{-1/2}}{\sqrt{2\pi}} e^{-t/2} = \frac{\frac{1}{2} e^{-t/2} (\frac{1}{2} t)^{-1/2}}{\Gamma(1/2)}$$

where $\sqrt{\pi} = \Gamma(1/2)$, so Y is gamma with parameters $\lambda = 1/2$ and $r = 1/2$.

10. (a) $F(x) = \alpha F_d(x) + \beta F_c(x)$, where

$$\begin{aligned} \alpha &= Pr\ \{\text{X is discrete}\} &&= Pr\{X = 1, X = -1\} \\ &= \Phi(-1) + (1 - \Phi(1)) &&= 2(1 - \Phi(1)) = 0.3174. \\ \beta &= 1 - \alpha = 0.6826. \end{aligned}$$

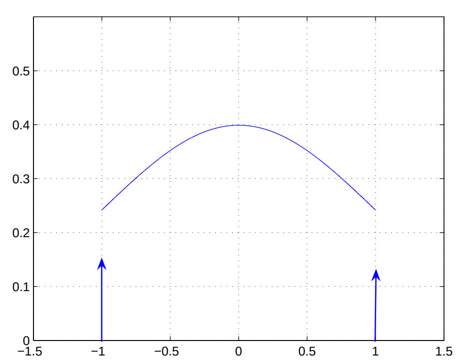
and

$$F_d(x) = \begin{cases} 0 & x < -1 \\ 0.5 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$F_c(x) = \begin{cases} 0 & x < -1 \\ \frac{\Phi(x) - 0.1587}{0.6826} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

(b)

$$\begin{aligned} f(x) &= F'(x) \\ &= 0.3174[0.5\ \delta(x+1) + 0.5\ \delta(x-1)] + 0.6826(\tfrac{\phi(x)}{0.6826})[u(x+1) - u(x-1)]. \end{aligned}$$



(c)

$$\begin{aligned} P(X < 0) &= \Phi(0) = 1/2. \\ P(X \leq 0) &= \Phi(0) = 1/2. \\ P(X < 1) &= \Phi(1). \\ P(X \leq 1) &= P(X < 1) + P(X = 1) = \Phi(1) + 1 - \Phi(1) = 1. \end{aligned}$$

(d)

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x)\ dx \\
&= \int_{-\infty}^{\infty} 0.1587[\delta(x+1)+\delta(x-1)]x\ dx \\
&\quad + \int_{-\infty}^{\infty} x\phi(x)[u(x+1)-u(x-1)]dx \\
&= 0 + \int_{-1}^{1} x\phi(x)dx \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= \int_{-\infty}^{\infty} x^2 f(x)\ dx \\
&= \int_{-\infty}^{\infty} 0.1587[\delta(x+1)+\delta(x-1)]x^2\ dx \\
&\quad + \int_{-\infty}^{\infty} x^2\phi(x)[u(x+1)-u(x-1)]dx \\
&= 2 * 0.1587 + \int_{-1}^{1} x^2\phi(x)dx \\
&= 0.3174 + \int_{-1}^{1} \phi(x)'' dx + \int_{-1}^{1} \phi(x)dx \\
&\approx 0.5
\end{aligned}
$$

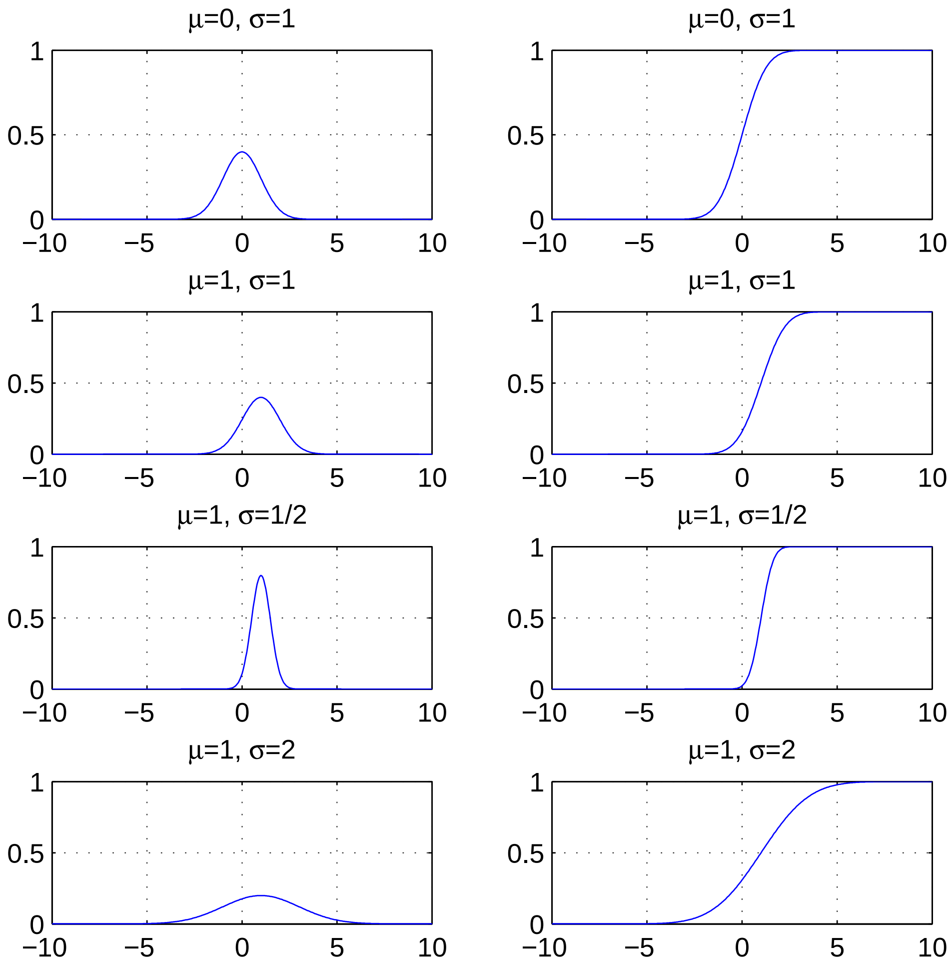


Figure 1: Question 2