ECSE 305A: Probability and Random Signals I
Problem Set 6
solutions

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1. (a) It is a continuous random variable, because its CDF is absolutely continuous.
   (b)
$$f(x) = \begin{cases} 0 & X < 0 \\ \frac{1}{4} & 0 \leq x < 4 \\ 0 & x \geq 4 \end{cases}$$
   (c)
$$P(X \geq 5) = 1 - P(X < 5) = 1 - F(5^-) = 1 - F(5) = 0$$
$$P(X < 0) = F(0^-) = F(0) = 0$$
$$P(X \leq 0) = F(0) = 0$$
$$P(\frac{1}{4} \leq X < 1) = \int_{1/4}^{1} \frac{1}{4} dx = 3/16$$
$$P(\frac{1}{4} \leq X \leq 1) = \int_{1/4}^{1} \frac{1}{4} dx = 3/16$$
$$P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - \int_{-\infty}^{1/2} \frac{1}{4} dx = 7/8$$

2. $P(X > 15) = \int_{15}^{\infty} \frac{1}{15} e^{-x/15} dx = \frac{1}{e}$. Thus, the answer is
$$\sum_{i=4}^{8} \binom{8}{i} (\frac{1}{e})^i (1 - \frac{1}{e})^{8-i} = 0.3327$$

3. Let G and g be the distribution and density functions of Z, respectively. For $-\pi/2 < z < \pi/2$,

$$\begin{aligned} G(z) &= P(\arctan X \le z) = P(X \le \tan z) = \int_{-\infty}^{\tan z} \frac{1}{\pi(1+x^2)} dx \\ &= [\tfrac{1}{\pi}\arctan x]_{-\infty}^{\tan z} = \tfrac{1}{\pi}z + \tfrac{1}{2}. \end{aligned}$$

Thus,

$$g(z) = \begin{cases} \frac{1}{\pi} & -\pi/2 < z < \pi/2 \\ 0 & \text{elsewhere.} \end{cases}$$

4. Let G and g be the distribution and density functions of t, respectively. Then

$$\begin{aligned} g(t) &= \sum_i f(x_i)|\tfrac{dx_i}{dy}| \\ &= \begin{cases} e^{-t} + \frac{1}{t^2}e^{-1/t} & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

Therefore,

$$G(t) = \int_{-\infty}^{t} g(t)dt = \begin{cases} 0 & t \le 0 \\ 1 - e^{-t} + e^{-1/t} & 0 < t < 1 \\ 1 & t \ge 1. \end{cases}$$

5. For $i = 0, 1, 2, \ldots, n-1$,

$$P(\lfloor nX \rfloor = i) = P(i \le nX < i+1) = P(\frac{i}{n} \le X < \frac{i+1}{n}) = \frac{1}{n}.$$

$P(\lfloor nX \rfloor = i) = 0$, otherwise. Therefore, $\lfloor nX \rfloor$ is a random number from the set $\{0, 1, 2, \ldots, n-1\}$.

6. The length of the other side is given by $\sqrt{9^2 - x^2}$. Therefore, the expected value is:

$$E(\sqrt{81 - x^2}) = \int_2^4 \sqrt{81 - x^2} \cdot \frac{x}{6} dx.$$

Assume $u = 81 - x^2$,then $du = -2xdx$ and

$$E\left(\sqrt{81 - x^2}\right) = \frac{1}{12}\int_{65}^{77} \sqrt{u}du \approx 8.4.$$

7.

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{2} x e^{-|x|} dx = 0$$

because the integrand is an odd function.

$$E(X^2) = \int_{-\infty}^{\infty} \frac{1}{2} x^2 e^{-|x|} dx = \int_{0}^{\infty} x^2 e^{-|x|} dx = 2$$

because the integrand is an even function, the result is twice the integration from zero to infinity.

$$Var(X) = E(X^2) - E(X)^2 = 2.$$

8. $E(e^X) = \int_0^\infty e^x (3e^{-3x}) dx = \int_0^\infty 3e^{-2x} dx = 3/2.$