ECSE 305A: Probability and Random Signals I
Problem Set 5
solutions

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1. (a)$c(-1)^2 + c + 4c + 9c = 1 \longrightarrow c = 1/15 \quad \mu = 4/3$

   (b)$c * \sum_{x=1}^{\infty} \alpha^x = \frac{c\alpha}{1-\alpha} = 1 \longrightarrow c = \frac{1-\alpha}{\alpha} \quad if|\alpha| < 1$
   $\mu = \sum_{x=1}^{\infty} cx\alpha^x = \frac{c\alpha}{(1-\alpha)^2} = \frac{1}{1-\alpha}$

   (c)

$$\begin{aligned}
\sum_{x=0}^{2n} c|n-x| &= \sum_{x=0}^{n} c(n-x) + \sum_{x=n+1}^{2n} c(x-n) \\
&= (n+1)cn + c\sum_{x=0}^{n} x - n(cn) + c\sum_{n+1}^{2n} x \\
&= cn(n+1) \\
&= 1 \\
&\longrightarrow c = 1/(n(n+1))
\end{aligned}$$

$$\begin{aligned}
\mu &= \sum_{x=0}^{2n} cx|n-x| \\
&= \sum_{x=0}^{n} cx(n-x) + \sum_{x=n+1}^{2n} cx(x-n) \\
&= \frac{1}{n(n+1)}\left(\frac{n^2(n+1)}{2} - \frac{2n(n+1)(2n+1)}{6} + \frac{2n(2n+1)(4n+1)}{6} - \frac{n(2n-n)(3n+1)}{2}\right) \\
&= n
\end{aligned}$$

2. (a) $E(X) = \sum_{x=1}^{5} x \cdot x/15 = 11/3$; $E(X^2) = \sum_{x=1}^{5} x^2 \cdot x/15 = 15$
   $Var(X) = E(X^2) - (E(X))^2 = 14/9$.

   (b) $p_Y(y) = \begin{cases} 1/15 + 5/15 = 6/15 & y = 5 \\ 2/15 + 4/15 = 6/15 & y = 8 \\ 3/15 & y = 9 \\ 0 & else \end{cases}$

(c)

$$E(Y) = 5 \cdot 6/15 + 8 \cdot 6/15 = 9 \cdot 3/15 = 21/3.$$

or

$$E(Y) = E(X(6-X)) = E(6X)-E(X^2) = 6E(X)-E(X^2) = 6\cdot 11/3-15 = 21/3.$$

3.

$$\begin{aligned}
E(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=1}^{n} x^2 \frac{n!}{x!(n-x)!} p^x q^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n(n-1)!}{(x-1)!(n-1-(x-1))!} p\, p^{x-1} q^{n-1-(x-1)} \\
&= \sum_{y=0}^{m} (y+1) \frac{nm!}{(y)!(m-y)!} p\, p^y q^{m-y} \\
&= np(\sum_{y=0}^{m} \frac{y\, m!}{y!(m-y)!} p^y q^{m-y} + \sum_{y=0}^{m} \frac{y!}{m!(y-m)!} p^y q^{m-y}) \\
&= np(mp+1) \\
&= np((n-1)p+1) \\
&= np(np+q).
\end{aligned}$$

$$Var(X) = E(X^2) - \mu^2 = n^2p^2 + npq - n^2p^2 = npq.$$

4. If X denotes the number of persons who decide correctly among a three-person jury, then X is a binomial random variable with parameters $(3, p)$.
Probability that a three-person jury decides correctly is:

$$\begin{aligned}
P(X \geq 2) &= P(X=2) + P(X=3) \\
&= \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3(1-p)^0 \\
&= 3p^2(1-p) + p^3 \\
&= 3p^2 - 2p^3.
\end{aligned}$$

The condition that a three-person jury is preferable to a single juror is if and only if:

$$\begin{aligned}
3p^2 - 2p^3 &> p \\
3p - 2p^2 - 1 &> 0 \\
2(1-p)(p-1/2) &> 0
\end{aligned}$$

Since $1-p > 0$ then $p-1/2$ must be positive, which results in $p > 1/2$. Therefore, the three-person jury is preferable if $p > 1/2$. In case $p < 1/2$, the single juror is preferable and if $p = 1/2$ there is no difference.

5. (a)

$$1 - 0.55 * \sum_{i=0}^{5} 0.45^i = 0.0083$$

(b)

$$0.55 * 0.45^3 * 0.55 * (\sum_{i=0}^{2} 0.45^i) = 0.0456$$

6. (a) If $S_n = \sum_{k=1}^{\infty} k\rho^k$, then

$$\begin{array}{rcl} S_n - \rho S_n & = & \rho + \rho^2 + \rho^3 + \cdots \\ & = & \frac{\rho}{1-\rho} \\ & \Longrightarrow & S_n = \frac{\rho}{(1-\rho)^2} \end{array}$$

(b) If $S_n = \sum_{k=1}^{\infty} k^2\rho^k$, then

$$\begin{array}{rcl} S_n - \rho S_n & = & \sum_{k=1}^{\infty}((k)^2 - (k-1)^2)\rho^k \\ & = & \sum_{k=1}^{\infty}((k+k-1)(k-k+1))\rho^k \\ & = & 2\sum_{k=1}^{\infty} k\rho^k - \sum_{k=1}^{\infty} \rho^k \\ & = & \frac{2\rho}{(1-\rho)^2} - \frac{\rho}{1-\rho} \\ & \Longrightarrow & S_n = \frac{\rho(1+\rho)}{(1-\rho)^3} \end{array}$$

7. (a) If X denotes the number of the faculty, who was born on 1st of January, then X is Binomial RV with parameters (n=45,p=1/365). Then,

$$p(x) = P(X = x) = \begin{cases} \binom{45}{0}(1/365)^0(1-1/365)^{45-0} \approx 0.88 & x = 0 \\ \binom{45}{1}(1/365)^1(1-1/365)^{45-1} \approx 0.10 & x = 1 \\ \binom{45}{2}(1/365)^2(1-1/365)^{45-2} \approx 0.006 & x = 2 \\ \binom{45}{3}(1/365)^3(1-1/365)^{45-3} \approx 0.0002 & x = 3 \end{cases}$$

(b) We can approximate the Binomial RV with a Poisson RV with $\lambda = np = 45 * 1/365 \approx 0.12$. Then,

$$p(x) = P(X = x) = \begin{cases} \frac{e^{-0.12}0.12^0}{0!} \approx 0.88 & x = 0 \\ \frac{e^{-0.12}0.12^1}{1!} \approx 0.10 & x = 1 \\ \frac{e^{-0.12}0.12^2}{2!} \approx 0.006 & x = 2 \\ \frac{e^{-0.12}0.12^3}{3!} \approx 0.0002 & x = 3 \end{cases}$$

8. Let X denotes the number of errors on the specific page, then X is a Poisson random variable with $E(X) = 1/5$. Therefore, $\lambda = 1/5$ and $P(X = n) = \frac{(1/5)^n e^{-1/5}}{n!}$. Hence, $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1/5} \approx 0.18$

9. (a) The probability of reaching the decision on a certain round of coin tossing is:

$$\binom{3}{2}p^2q + \binom{3}{2}q^2p = 3pq(p+q) = 3pq.$$

The probability of not reaching the decision is $1-3pq$. If X denotes the number of tosses until they reach the decision, then X is a geometric random variable with the parameter $3pq$. Therefore,

$$P(X < n) = 1 - P(X \geq n) = 1 - (1 - 3pq)^{n-1}.$$

(b)We want to find the minimum number (n) of tosses so that:

$$\begin{aligned} P(X \leq n) &\geq 0.95 \\ 1 - P(X > n) &\geq 0.95 \\ P(X > n) &\leq 0.05. \end{aligned}$$

But, $P(X > n) = (1 - 3pq)^n = (1 - 3/4)^n = (1/4)^n$. Thus, $(1/4)^n \leq 0.05 \Rightarrow n \geq 2.16 \Rightarrow$ the smallest n is 3.

10. (a) $R_X = \{r, r+1, r+2, \cdots\}$.

(b)The probability that the rth success occurs in the nth trial is equal to the probability that exactly $r-1$ success happen in the $n-1$ number of trials and the nth trial is a success. Therefore,

$$\begin{aligned} P(X = n) &= \tbinom{n-1}{r-1}p^{r-1}(1-p)^{(n-1)-(r-1)} \times p \\ &= \tbinom{n-1}{r-1}p^r(1-p)^{n-r}. \end{aligned}$$

Hence, the PMF for the negative binomial RV is:

$$p(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r} where \;\; 0 < p < 1 \;\; and \;\; x = r, r+1, \cdots.$$

(c) Let X denotes the number of the games until Bill wins and Y the number of the games until Monica wins. Then X and Y are negative binomial RVs with parameters $(5, 0.42)$ and $(5, 1 - 0.42 = 0.58)$, respectively. The probability that the series ends in 8 games is:

$$P(X = 7) + P(X = 7) = \binom{7}{4}0.42^5 0.58^3 + \binom{7}{4}0.58^5 0.42^3 \approx 0.089 + 0.17 \approx 0.25.$$