ECSE 305A: Probability and Random Signals I
Problem Set 4
solutions

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October 5, 2006

1. (a) X is a random variable of mixed type since it is continuous except for discontinuity at 0 and at 1.
(b)
$$P(X \geq 5) = 1 - P(X < 5) = 1 - F(5^-) = 0$$
$$P(X < 0) = F(0^-) = 0$$
$$P(X \leq 0) = F(0) = \frac{1}{4}$$
$$P(\frac{1}{4} \leq X < 1) = F(1^-) - F(\frac{1}{4}^-) = \frac{3}{16}$$
$$P(\frac{1}{4} \leq X \leq 1) = F(1) - F(\frac{1}{4}^-) = \frac{11}{16}$$
$$P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - F(\frac{1}{2}) = \frac{5}{8}$$

2. (a) It can be seen that CDF $F(x)$ is flat except for a finite number of jumps, so X is a discrete RV.
(b)
$$p(x) = \begin{cases} x < -2 & 0 \\ x = -2 & F(-2) - F(-2^-) = 1/4 - 0 = 1/4 \\ x = -1 & F(-1) - F(-1^-) = 1/4 - 1/4 = 0 \\ x = 0 & F(0) - F(0^-) = 1/2 - 1/4 = 1/4 \\ x = 1 & F(1) - F(1^-) = 3/4 - 1/2 = 1/4 \\ x = 2 & F(2) - F(2^-) = 1 - 3/4 = 1/4 \\ x > 2 & 0 \end{cases}$$

(c) Assuming $Y = X^2$ then $R_y = \{0, 1, 4\}$

$$p(y) = \begin{cases} y < 0 & 0 \\ y = 0 & \sum_{x^2=0} p(x) = p(x=0) = 1/4 \\ y = 1 & \sum_{x^2=1} p(x) = p(x=1) + p(x=-1) = 1/4 + 0 = 1/4 \\ y = 4 & \sum_{x^2=4} p(x) = p(x=2) + p(x=-2) = 1/4 + 1/4 = 1/2 \\ y > 4 & 0 \end{cases}$$

3. Let p be the probability function of X and F be its distribution function. We have

$$p(i) = (\frac{5}{6})^{i-1}(\frac{1}{6}), \qquad i = 1, 2, 3, \ldots$$

$F(x) = 0$ for $x < 1$. If $x \geq 1$, for some positive integer n, $n \leq x < n+1$, and we have that

$$\begin{aligned} F(x) &= \textstyle\sum_{i=1}^{n} (\frac{5}{6})^{i-1}(\frac{1}{6}) \\ &= \frac{1}{6} * \frac{1-(\frac{5}{6})^n}{1-\frac{5}{6}} \\ &= 1 - (\frac{5}{6})^n \end{aligned}$$

Hence

$$y = \begin{cases} 0 & \text{if } x < 1 \\ 1 - (\frac{5}{6})^n & \text{if } n \leq x < n+1, \quad n = 1, 2, 3, \ldots \end{cases}$$

4.

$$\begin{aligned} \textstyle\sum_{k=0}^{\infty} P(X > k) &= P(X > 0) & +P(X > 1) & & +P(X > 2) + \cdots \\ &= P(X = 1) & +P(X = 2) & & +P(X = 3) + \cdots \\ & & +P(X = 2) & & +P(X = 3) + \cdots \\ & & & & +P(X = 3) + \cdots \\ &= 1 \cdot P(X = 1) & +2 \cdot P(X = 2) & & +3 \cdot P(X = 3) + \cdots \\ &= E[X]. \end{aligned}$$

5. There are two ways to solve this problem:
(a)

$$\begin{aligned} E(Y) &= \textstyle\sum_{k=0}^{\infty} Y * p_k \\ &= \textstyle\sum_{k=1}^{\infty} (k-1)p_k \\ &= \textstyle\sum_{k=0}^{\infty} (k-1)p_k + p_0 \\ &= E(X) - 1 + p_0 \end{aligned}$$

$$
\begin{aligned}
E(Y^2) &= \textstyle\sum_{k=0}^{\infty} Y^2 * p_k \\
&= \textstyle\sum_{k=1}^{\infty} (k-1)^2 p_k \\
&= \textstyle\sum_{k=0}^{\infty} (k-1)^2 p_k - p_0 \\
&= \textstyle\sum_{k=0}^{\infty} (k^2 - 2k + 1) p_k - p_0 \\
&= E(X^2) - 2E(X) + 1 - p_0
\end{aligned}
$$

(b) (Optional) Define Moment Function as:

$$\Gamma(z) = E(z^n) = \sum_{n=-\infty}^{\infty} p_n z^n$$

Differentiating $\Gamma(z)$ k times, we obtain

$$\Gamma^{(k)}(z) = E[n(n-1)(n-2)\cdots(n-k+1)z^{n-k}]$$

With $z = 1$, this yields

$$\Gamma^{(k)}(1) = E[n(n-1)(n-2)\cdots(n-k+1)]$$

In particular, that $\Gamma(1) = 1$, $\Gamma'(1) = E(n)$ and $\Gamma''(1) = E(n^2) - E(n)$
In this problem, it is obvious that $\Gamma_y(z) = p_0 + z^{-1}[\Gamma_x(z) - p_0]$.
Therefore

$$\Gamma_y'(z) = z^{-1}(z\Gamma_x'(z) - \Gamma_x(z) + p_0)$$

and

$$\Gamma_y''(z) = z^{-1}(z^2\Gamma_x''(z) - z\Gamma_x'(z) + \Gamma_x(z) - p_0)$$

Let $z = 1$ and we have $\Gamma(1) = 1$, $\Gamma'(1) = E(n)$ and $\Gamma''(1) = E(n^2) - E(n)$. Therefore

$$E(Y) = E(X) - 1 + p_0, \qquad E(Y^2) = E(X^2) - 2E(X) + 1 - p_0$$

6. Let X be the number of children they should continue to have until they have one of each sex. Let B be the event that the first child is a boy. Then

$$
\begin{aligned}
P(X = i) &= P(X = i|B)P(B) + P(X = i|B^c)P(B^c) \\
&= (\tfrac{1}{2})^{i-2}(\tfrac{1}{2}) \cdot \tfrac{1}{2} + (\tfrac{1}{2})^{i-2}(\tfrac{1}{2}) \cdot \tfrac{1}{2} \\
&= (\tfrac{1}{2})^{i-1}
\end{aligned}
$$

So

$$E(X) = \sum_{i=2}^{\infty} i(\frac{1}{2})^{i-1} = -1 + \sum_{i=1}^{\infty} i(\frac{1}{2})^{i-1} = 3$$

7. (a)Let X be the number of trials required to open the door. Clearly,

$$P(X = x) = (1 - \frac{1}{n})^{x-1}\frac{1}{n}, \qquad x = 1, 2, 3, \ldots$$

Thus

$$\begin{aligned} E(X) &= \textstyle\sum_{x=1}^{\infty} x(1 - \frac{1}{n})^{x-1}\frac{1}{n} \\ &= \frac{1}{n}\frac{1}{[1-(1-\frac{1}{n})]^2} \\ &= n \end{aligned}$$

To calculate $Var(X)$, first we find $E(x^2)$. We have

$$\begin{aligned} E(X^2) &= \textstyle\sum_{x=1}^{\infty} x^2(1 - \frac{1}{n})^{x-1}\frac{1}{n} \\ &= \frac{1}{n}\textstyle\sum_{x=1}^{\infty} x^2(1 - \frac{1}{n})^{x-1} \\ &= \frac{1}{n}\frac{1+(1-\frac{1}{n})}{[1-(1-\frac{1}{n})]^3} \\ &= 2n^2 - n \end{aligned}$$

Therefore

$$Var(X) = (2n^2 - n) - n_2 = n(n-1)$$

(b) Let A_i be the event that on the ith trial the door opens. Let X be the number of trials required to open the door. Then

$$\begin{aligned} P(X = k) &= P[A_1^c \cap A_2^c \cap \ldots \cap A_{k-1}^c \cap A_k] \\ &= P(A_k|A_1^c \cap A_2^c \cap \ldots \cap A_{k-1}^c) * \\ &\quad P(A_{k-1}^c|A_1^c \cap A_2^c \cap \ldots \cap A_{k-2}^c) * \\ &\quad \cdots * P(A_2^c|A_1^c) * P(A_1^c) \\ &= \frac{1}{n-k+1}\textstyle\prod_{i=n-k+2}^{n} \frac{i-1}{i} \\ &= \frac{1}{n}. \end{aligned}$$

$$E(X) = \sum_{i=1}^{n} i * \frac{1}{n} = \frac{n+1}{2}$$

$$E(X^2) = \sum_{i=1}^{n} i^2 * \frac{1}{n} = \frac{(n+1)(2n+1)}{6}$$

$$Var(X) = \frac{(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 = \frac{n^2-1}{12}$$