ECSE 305A: Probability and Random Signals I
Problem Set 3
solutions

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1. Let A be the event that during this period he has hiked in Westmount Park at least once. Let B be the event that during this period he has hiked in this park at least twice. We have

$$P(B|A) = \frac{P(B)}{P(A)},$$

where

$$P(A) = 1 - \frac{5^{10}}{6^{10}} = 0.838$$

and

$$P(B) = 1 - \frac{5^{10}}{6^{10}} - \frac{10 * 5^9}{6^{10}} = 0.515$$

So the answer is $\frac{0.515}{0.838} = 0.615$

2. (a)For $1 \leq n \leq 39$, let E_n be the event that none of the first $n-1$ cards is a heart or the ace of spades. Let F_n be the event that the nth card drawn is the ace of spades. Then the event of "no heart before the ace of spades" is $\cup_{n=1}^{39} E_n F_n$. Clearly, $\{E_n F_n, 1 \leq n \leq 39\}$ forms a sequence of mutually exclusive events. Hence,

$$\begin{aligned} P(\cup_{n=1}^{39}(E_n \cap F_n)) &= \textstyle\sum_{n=1}^{39} P(E_n \cap F_n) \\ &= \textstyle\sum_{n=1}^{39} P(E_n)P(F_n|E_n) \\ &= \textstyle\sum_{n=1}^{39} \frac{\binom{38}{n-1}}{\binom{52}{n-1}} * \frac{1}{53-n} \\ &= \frac{1}{14}. \end{aligned}$$

(b) Let the event D_i be each placement that the ace of spades is placed before all the hearts. There are $\binom{52}{14}$ of these mutually exclusive events.

In each event 14 positions is chosen and ace of the spades is placed in the first place and then the hearts are placed in 13! different ways in chosen places and at last the rest of the cards can be placed in 38! ways in remaining places. So the probability of event A is given by:

$$\begin{aligned} P(A) &= \frac{\binom{52}{14}13!38!}{52!} \\ &= \frac{1}{14}. \end{aligned}$$

3. Let A_n be the event that Don loses the first n games. By the law of multiplication,

$$P(A_n) = \frac{2}{3} * \frac{3}{4} * \frac{4}{5} * \cdots * \frac{n+1}{n+2} = \frac{2}{n+2}$$

Now since $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots A_n \supseteq A_{n+1} \supseteq \cdots$, Hence

$$P(\cap_{i=1}^{\infty}) = \lim_{n\to\infty} P(A_n) = 0$$

4. (a)

$$\begin{aligned} P(AB|C) &= \frac{P(ABC)}{P(C)} \\ &= \frac{P(ABC)}{P(C)} \times \frac{P(BC)}{P(BC)} \\ &= \frac{P(ABC)}{P(BC)} \times \frac{P(BC)}{P(C)} \\ &= P(A|BC)P(B|C). \end{aligned}$$

(b)

$$\begin{aligned} P(A|C) &= \frac{P(AC)}{P(C)} \\ &= \frac{P(AC\cap S)}{P(C)} \\ &= \frac{P(AC\cap(B_1\cup\cdots\cup B_n))}{P(C)} \\ &= \frac{P((ACB_1)\cup\cdots\cup P(ACB_n))}{P(C)} \\ &= \sum_{i=1}^{n} \frac{P(ACB_i)}{P(C)} \times \frac{P(B_iC)}{P(B_iC)} \\ &= \sum_{i=1}^{n} P(A|B_iC)P(B_i|C) \end{aligned}$$

5. Let G be the event that the randomly selected child is a girl, A be the event that she has an older sister, and O, M, and Y be the events that she is the oldest, the middle, and the youngest child of the family, respectively. We have that

$$\begin{aligned} P(A|G) &= P(A|G\cap O)P(O|G) + P(A|G\cap M)P(M|G) + \\ &\quad P(A|G\cap Y)P(Y|G) \\ &= 0 * \tfrac{1}{3} + \tfrac{1}{2} * \tfrac{1}{3} + \tfrac{3}{4} * \tfrac{1}{3} = \tfrac{5}{12} \end{aligned}$$

6. $\frac{0.15*0.25}{0.15*0.25+0.85*0.75} = 0.056$

7. (a) $1-(\frac{1}{2})^n$
(b) $\binom{n}{k}(\frac{1}{2})^n$
(c) Let A_n be the event of getting n heads in the first n flips. We have

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots A_n \supseteq A_{n+1} \supseteq \cdots$$

The event of getting heads in all of the flips indefinitely is $\cap_{n=1}^{\infty} A_n$. By the continuity property of probability function, its probability is

$$P(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} (\frac{1}{2})^n = 0$$

8. The events $A_1, A_2, \ldots, A_n$ are mutually independent if any $k < n$ of them are independent and

$$P(A_1, A_2, \cdots, A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

For a certain k, $\binom{n}{k}$ equations are needed. Then the total number of equations is

$$\begin{aligned} & \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1} + \binom{n}{n} \\ = & \sum_{i=0}^{n} \binom{n}{i} - \binom{n}{0} - \binom{n}{1} \\ = & 2^n - 1 - n = 2^n - (1+n) \end{aligned}$$

9. (a) Let H, H_1, and H_2 denote the events that the mother, the first son, and the second son are hemophilic, respectively. We have that

$$P(H_1) = P(H_1|H)P(H) + P(H_1|H^c)P(H^c) = \frac{1}{2} * \frac{1}{4} + 0 * \frac{3}{4} = \frac{1}{8}$$

(b) Similarly, $P(H_2) = \frac{1}{8}$.
(c) Note that H_1 and H_2 are *conditionally independent* given H. That is, if we are given that the mother is hemophilic, knowledge about one son being hemophilic does not change the chance of the other son being hemophilic. However, H_1 and H_2 are not independent. This is because if we know that one son is hemophilic, the mother is hemophilic and therefore with probability $\frac{1}{2}$ the other son is also hemophilic.
To calculate $P(H_1^c \cap H_2^c)$, the probability the none of her sons is hemophiliac, we condition on H again:

$$P(H_1^c \cap H_2^c) = P(H_1^c \cap H_2^c|H)P(H) + P(H_1^c \cap H_2^c|H^c)P(H^c)$$

Clearly, $P(H_1^c \cap H_2^c | H^c)$=1. To find $P(H_1^c \cap H_2^c | H)$, we use the fact that H_1 and H_2 are conditionally independent given H:

$$P(H_1^c \cap H_2^c | H) = P(H_1^c | H) P(H_2^c | H) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

Thus,

$$P(H_1^c \cap H_2^c) = \frac{1}{4} * \frac{1}{4} + 1 * \frac{3}{4} = \frac{3}{16}$$