ECSE 305A: Probability and Random Signals I
Problem Set 2
solutions

McGill University

September 28, 2006

1. (a) {*ppp, ppf, pfp, fpp, pff, fpf, ffp, fff*};
   (b) $Z_F = \{ppf, pff, fpf, fff\}$,$X_P = \{ppp, ppf, pfp, pff\}$;
   (c) No;
   (d)
   $$\begin{aligned} P(Z_F \cup X_P) &= P(Z_F) + P(X_P) - P(Z_F \cap X_P) \\ &= \tfrac{4}{8} + \tfrac{4}{8} - \tfrac{2}{8} \\ &= \tfrac{3}{4} \end{aligned}$$
   (e) $C = \{ppp, ppf, pfp, fpp\}$, $D = \{pff, fpf, ffp, fff\}$;
   (f) Yes;
   (g)
   $$\begin{aligned} P(C \cup D) &= P(C) + P(D) \\ &= \tfrac{4}{8} + \tfrac{4}{8} \\ &= 1. \end{aligned}$$

2. (a)
   $$\begin{aligned} & \text{Probability that exactly two of the events happen} \\ = \; & P(AB - C) + P(AC - B) + P(BC - A) \\ = \; & P(AB) - P(ABC) + P(AC) - P(ABC) + P(BC) - P(ABC) \\ = \; & P(AB) + P(AC) + P(BC) - 3P(ABC). \end{aligned}$$

   In the above derivation we have used the theorem that $P(A - B) = P(A) - P(AB)$.
   (b) S1: false
   $A = B = C$ and $P(A) = P(B) = P(C) = 1/3$ then $P(A) + P(B) + P(C) = 1$ but A, B and C are not mutually exclusive.
   S2: false

$A = B = C = S$ then $P(A \cup B \cup C) = 1$ but A, B and C are not mutually exclusive.

3. (a) $S = \{(x, y) : 0 \leq x \leq y \leq 24\}$;
   (b) $A = \{(x, y) : 0 \leq x \leq 9, 9 \leq y \leq 24\}$, $\frac{15}{32}$;
   (c) $B = \{(x, y) : y - x < 12\}$,$\frac{3}{4}$;
   (d) At 9 o'clock the student is sleep and he is asleep more than he is awake,$\frac{33}{64}$

4. (a) $\frac{3}{4}$, $\frac{1}{4}$, 0;
   (b) 1, $\frac{5}{8}$.

5. The number of the runs with no successive heads is equal to F_{n+2} (F =Fibonacci number). For the proof we use mathematical induction, the case for length 2 is evident. Considering the runs with no successive heads of length n, such runs end on head (H) or tail (T). If they end in T, they can be generated by adding a T to the end of all such runs of length $n-1$. If they end in H, there must be a T before that, so they are generated of such runs of length $n-2$, by adding a TH at the end. So number of such runs is equal to the addition of runs with no successive heads of length $n-1$ and $n-2$, which is exactly the relation between the Fibonacci numbers, where $F_n = F_{n-1} + F_n$.
   $P = \frac{F_{n+2}}{2^n} = \frac{1}{10*2^{2n}}[(5+3\sqrt{5})(1+\sqrt{5})^n + (5-3\sqrt{5})(1-\sqrt{5})^n]$.

6. (a) $\frac{n!}{m^n}$, (b) $\frac{n!(m-1)!}{(m+n-1)!}$, (c) $\frac{n!(m-n)!}{m!}$.

7. (a) $\frac{20!}{39*37*35*...*5*3*1} = 7.61 * 10^{-6}$,
   (b)$\frac{1}{39*37*35*...*5*3*1} = 3.13 * 10^{24}$.

8. $\frac{\binom{13}{2}\binom{13}{3}\binom{13}{4}\binom{13}{5}}{\binom{52}{14}}$

9. If A_i denote the event that the envelope i is empty, we want to find the probability $P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c)$.

   $$P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c) = 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6)$$

   Using the extension of *Theorem 3.4* we can compute the union of the events.

$$
\begin{aligned}
P(\cup_{i=1}^{n} A_i) &= \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(A_i A_j) \\
&+ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} P(A_i A_j A_k) \\
&- \cdots + (-1)^{n-1} P(A_1 A_2 \cdots A_n)
\end{aligned}
$$

$$
\begin{aligned}
P(\cup_{i=1}^{6} A_i) &= \sum_{i=1}^{6} P(A_i) - \sum_{i=1}^{5} \sum_{j=i+1}^{6} P(A_i A_j) \\
&+ \sum_{i=1}^{4} \sum_{j=i+1}^{5} \sum_{k=j+1}^{6} P(A_i A_j A_k) \\
&- \cdots + (-1)^{5} P(A_1 A_2 \cdots A_6) \\
&= \binom{6}{1}\frac{5^{10}}{6^{10}} - \binom{6}{2}\frac{4^{10}}{6^{10}} + \binom{6}{3}\frac{3^{10}}{6^{10}} - \binom{6}{4}\frac{2^{10}}{6^{10}} + \binom{6}{5}\frac{1^{10}}{6^{10}} - \binom{6}{6}\frac{0^{10}}{6^{10}}
\end{aligned}
$$

$$
\begin{aligned}
P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c \cap A_6^c) &= 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6) \\
&= 0.2718.
\end{aligned}
$$

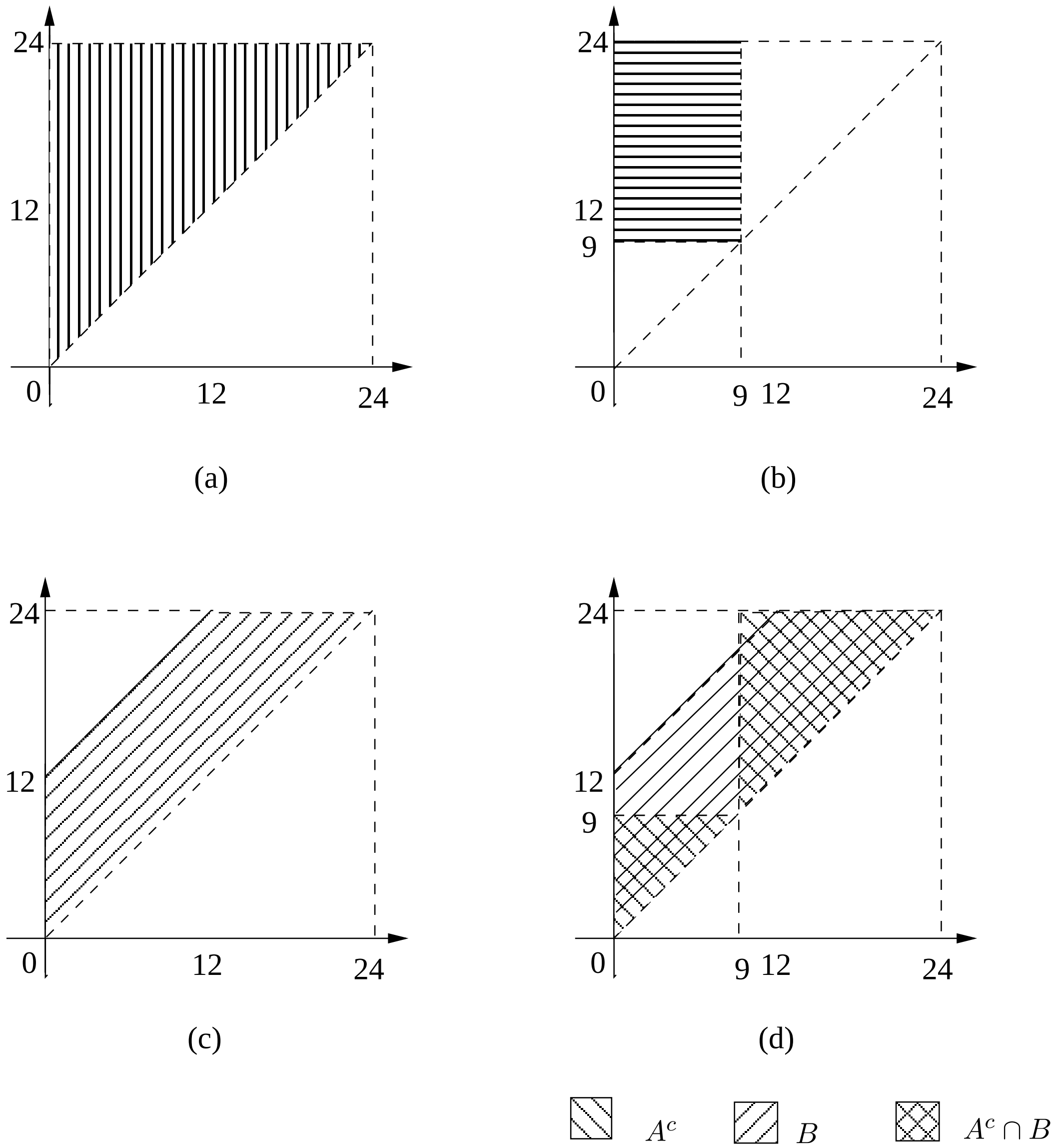

Figure 1: Question 3