ECSE 305A: Probability and Random Signals I
Problem Set 1
solutions

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1. (a) C and E;
   (b) D and E;
   (c) A, B and D;
   (d) None.

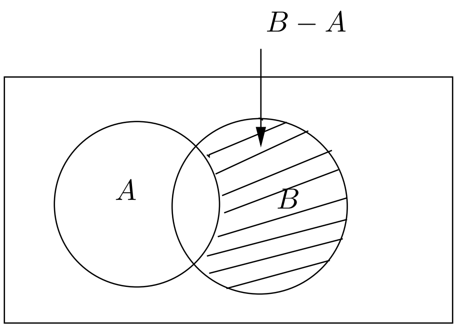
2. A: infinite and countable,
   B: infinite and uncountable,
   C: infinite and countable,
   D: finite,
   E: finite (empty set is finite).

3. (a)

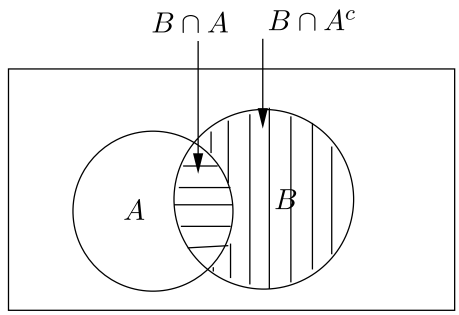
$$\begin{aligned} A \cup (B - A) &= A \cup (B \cap A^c) \\ &= (A \cup B) \cap (A \cup A^c) \\ &= (A \cup B) \cap S \\ &= A \cup B \end{aligned}$$

   (b)

$$\begin{aligned} (A \cap B) \cup (A^c \cap B) &= (B \cap A) \cup (B \cap A^c) \\ &= B \cap (A \cup A^c) \\ &= B \cap S \\ &= B \end{aligned}$$



$A \cup B = A \cup (B - A)$



$B = (A \cap B) \cup (A^c \cap B)$

4.

$$\lim_{i\to\infty} \frac{1}{i} = 0$$

Thus
A_i is decreasing, $\lim_{i\to\infty} A_i = \{0\}$;
B_i is decreasing, $\lim_{i\to\infty} B_i = \{0\}$;
C_i is increasing, $\lim_{i\to\infty} C_i = (0,1)$;
D_i is increasing, $\lim_{i\to\infty} D_i = [0,1)$.

5. (a) There are 3 ways to go from A to B and 2 ways to go from B to C; hence, $n = 3 \cdot 2 = 6$;
(b) There are 6 ways to go from A to C by way of B and 6 ways to return. Thus, $n = 6 \cdot 6 = 36$;
(c) The person will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

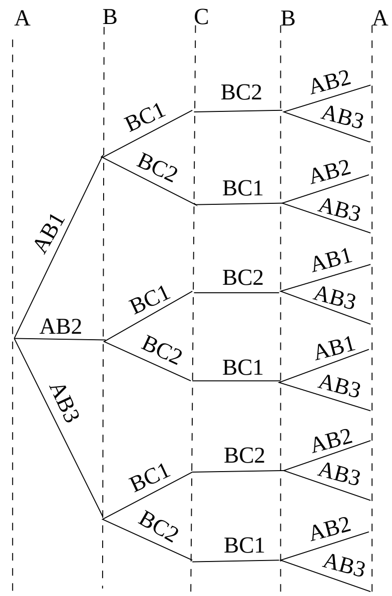
$$A \to B \to C \to B \to A$$

There are 3 ways to go from A to B and 2 ways to go from B to C. Since a bus line is not to be used more than once, there are only 1 ways to go from C back to B and only 2 ways to go from B back to A. Enter these number above the corresponding arrows as follows:

$$A \xrightarrow{3} B \xrightarrow{2} C \xrightarrow{1} B \xrightarrow{2} A$$

Thus, $n = 3 \cdot 2 \cdot 1 \cdot 2 = 12$.

(d) Naming the bus lines from A to B ($AB1, AB2, AB3$) and bus lines from B to C ($BC1, BC2$), the tree diagram will be:



6. (a) $n = 6! = 720$;
(b) There are 2 ways to distribute them according to sex: BBBGGG or GGGBBB. In each case the boys can sit in $3! = 6$ ways and the grils can sit in $3! = 6$ ways. Thus, altogether, there are $2 \cdot 3! \cdot 3! = 72$ ways;
(c) There are 4 ways to distribute them according to sex: GGGBBB, BGGGBB, BBGGGB and BBBGGG. There are $4 \cdot 3! \cdot 3! = 144$ ways;
(d) When the group sits in a circle, all the circular shifts of each permutation are the same. For instance, numbering the boys and girls, permutation $B_1B_2B_3G_1G_2G_3$, $B_2B_3G_1G_2G_3B_1$, $B_3G_1G_2G_3B_1B_2$ and all other circular shifts of this sitting are the same. For each permutation we have 6 circular shifts, therefore the numbers calculated above must be divided by 6.

7. (a) This concerns combinations, not permutation, since order does not count in a committee. There are "12 choose 4" such committees. That is,

$$n = C(12, 4) = \binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495$$

(b) The 2 boys can be chosen from the 9 boys in $\binom{9}{2}$ ways and the 2 girls can be chosen from the 3 girls in $\binom{3}{2}$ ways. Thus,

$$n = \binom{9}{2}\binom{3}{2} = \frac{9 \cdot 8}{2 \cdot 1} \cdot \frac{3 \cdot 2}{2 \cdot 1} = 108$$

(c) The 3 boys can be chosen from the 9 boys in $\binom{9}{3}$ ways and the 1 girls can be chosen from the 3 girls in $\binom{3}{1}$ ways. Thus

$$n = \binom{9}{3}\binom{3}{1} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} \cdot \frac{3}{1} = 252$$

(d) There are 3 cases in this problem: 1 girl, 2 girls and 3 girls. We can put all the ways together. Thus

$$n = \binom{9}{3}\binom{3}{1} + \binom{9}{2}\binom{3}{2} + \binom{9}{1}\binom{3}{3} = 252 + 108 + 9 = 369$$

Or we can remove the ways that have no girls from the total ways without restrictions. Thus,

$$n = \binom{12}{4} - \binom{9}{4} = 495 - 126 = 369$$

8. This problem concerns permutations with repetitions.$n = \frac{9!}{2!2!2!} = 45360$, since there are 9 letters of which 2 are M, 2 are T and 2 are E. When the letter C and E are chosen for the first and last letter, we have 7 letters of which 2 are M and 2 are T, therefore $n = \frac{7!}{2!2!}$.

9. This problem concerns permutations. For the first person in the line we have 10 possible choices. Since the second person in the line must be from other nationality, there are 5 choices available. For the third person, considering the condition on the nationality and the fact that we have already chosen one person from this nationality (first person) we have 4 choices. Therefore, $n = 10 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 2880$.

10. for mathematical induction, we must first prove that the theorem holds when $n = 0$, that is $(x + y)^0 = 1 = \sum_{i=0}^{0} \binom{0}{i} x^{0-i} y^i$ which is true.

    Then assuming that the theorem holds for case n, it must be shown that it holds for $n+1$.

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= x\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i + y\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j \\
&= \sum_{i=0}^{n}\binom{n}{i}x^{n+1-i}y^i + \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^{j+1} \\
&= x^{n+1} + \sum_{i=1}^{n}\binom{n}{i}x^{n+1-i}y^i + \sum_{j=0}^{n}\binom{n}{i}x^{n-j}y^{j+1} \\
&= x^{n+1} + \sum_{i=1}^{n}\binom{n}{i}x^{n+1-i}y^i + \sum_{i=1}^{n+1}\binom{n}{i-1}x^{n-i+1}y^i \\
&= x^{n+1} + \sum_{i=1}^{n}\binom{n}{i}x^{n+1-i}y^i + \sum_{i=1}^{n}\binom{n}{i-1}x^{n-i+1}y^i + y^{n+1} \\
&= x^{n+1} + y^{n+1} + \sum_{i=1}^{n}[\binom{n}{i} + \binom{n}{i-1}]x^{n+1-i}y^i \\
&= x^{n+1} + y^{n+1} + \sum_{i=1}^{n}\binom{n+1}{i}x^{n+1-i}y^i \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}x^{n+1-i}y^i
\end{aligned}$$

In the proof we have used the $Theorem 2.9$ stating that $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$ in the lecture notes.

The expansion of $(x+y)^n$ is a polynomial with $n+1$ terms. For a certain term $x^{n-i}y^i$, the coefficient can be view as choosing i "y" from n "x+y". It's equal to $\binom{n}{i}$

Thus

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$