ECSE 305A: Probability and Random Signals I
Problem Set 11
solutions

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1. (a) It can easily be seen that

$$p(X=1, Y=-1) = 0 \neq P(X=1)P(Y=-1) = \frac{1}{4} \cdot \frac{17}{48}$$

Therefore, X and Y are dependent.

(b) See Figure 1.

2. (a)

$$\begin{aligned} F(x,y) &= \int_0^1 \int_0^1 f(x,y)dxdy \\ &= \int_0^1 \int_0^1 cx^p y^q dxdy \\ &= \int_0^1 c\, \frac{x^{p+1}}{p+1}\Big]_0^1 y^q dy \\ &= \frac{c}{p+1}\ \frac{y^{q+1}}{q+1}\Big]_0^1 \\ &= \frac{c}{(p+1)(q+1)} \\ &= 1. \\ &\Rightarrow c = (p+1)(q+1). \end{aligned}$$

(b)

$$\begin{aligned} E(X^k Y^l) &= c\int_0^1 \int_0^1 x^k y^l x^p y^q dxdy \\ &= c\int_0^1 x^{k+p}dx\ \int_0^1 y^{l+q}dy \\ &= \frac{p+1}{k+p+1} \cdot \frac{q+1}{l+q+1}. \end{aligned}$$

(c) Using the result in part(b), we have

$$\mu_X = E(X) = \frac{p+1}{p+2}$$

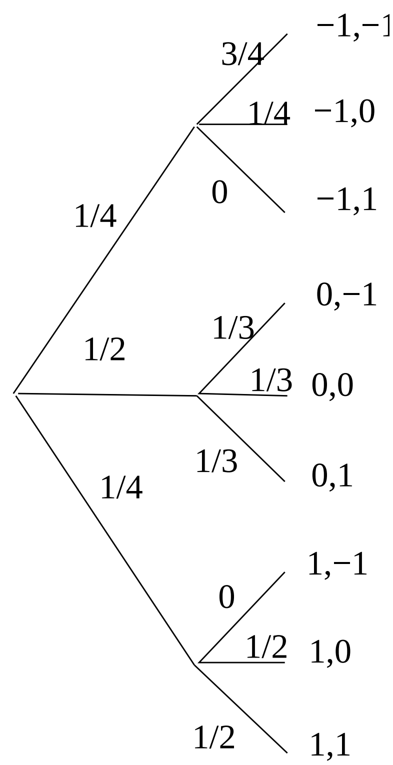

Figure 1: Question 1

$$\mu_Y = E(Y) = \frac{q+1}{q+2}$$

$$E(XY) = \frac{(p+1)(q+1)}{(p+2)(q+2)}.$$

Therefore,

$$Cov(X,Y) = E(XY) - \mu_X \mu_Y = 0.$$

As expected because two RV X and Y are independent.

(d)

$$Var(X+Y) = Var(X)+Var(Y)+2Cov(X,Y) = Var(X)+Var(Y)$$

Where,

$$Var(X) = E(X^2) - E^2(X) = \frac{p+1}{p+3} \cdot (\frac{p+1}{p+2})^2 = \frac{2}{4} \cdot (\frac{2}{3})^2 = \frac{1}{18}$$

$$Var(Y) = E(Y^2) - E^2(Y) = \frac{q+1}{q+3} \cdot (\frac{q+1}{q+2})^2 = \frac{2}{4} \cdot (\frac{2}{3})^2 = \frac{1}{18}$$

Therefore,

$$Var(X+Y) = \frac{1}{9}.$$

3. See Assignment 9.

4. (a)

$$\begin{aligned} F_W(w) &= P(W \le a) \\ &= \int\int_{0<x<1,\ 0<y<1,\ xy<a} dxdy \\ &= \int_0^a dx \int_0^1 dy + \int_a^1 dx \int_0^{a/x} dy \\ &= x]_0^a + a \ln x]_a^1 \\ &= a - a \ln a. \end{aligned}$$

(b)

$$f_w(w) = \frac{dP(W \le a)}{da} = 1 - \ln a - a/a = -\ln a$$

5. (a)

$$\begin{aligned} \psi(\omega) &= \sum_{x=0}^{\infty} \binom{n}{x} p^x q^{n-x} e^{-j\omega x} \\ &= \sum_{x=0}^{\infty} \binom{n}{x} (pe^{-j\omega})^x q^{n-x} \\ &= (pe^{-j\omega} + q)^n \end{aligned}$$

(b) Using the central limit theorem, we have

$$\begin{aligned} \psi_Y(\omega) &= \psi_1(\omega)\psi_2(\omega)\cdots\psi_k(\omega) \\ &= (pe^{-j\omega}+q)^{n_1}(pe^{-j\omega}+q)^{n_2}\cdots(pe^{-j\omega}+q)^{n_k} \\ &= (pe^{-j\omega}+q)^{n_1+n_2+\cdots+n_k} \end{aligned}$$

Therefore, $Y \sim B(n_1 + n_2 + \cdots + n_k, p)$.

6. (a)

$$\begin{aligned} R_y(\tau) &= h(\tau) * h(-\tau) * R_x(\tau) \\ &= h(\tau) * h(-\tau) * \delta(\tau) \\ &= h(\tau) * h(-\tau) \\ &= \frac{e^{-t}}{2}u(t) + \frac{e^t}{2}u(-t) \end{aligned}$$

(b)

$$\begin{aligned}
S_Y(\omega) &= H(\omega)H^*(\omega)S_X(\omega)\\
&= H(\omega)H^*(\omega)\\
&= \tfrac{1}{1+j\omega}(\tfrac{1}{1+j\omega})^*\\
&= \tfrac{1}{1+\omega^2}
\end{aligned}$$

7. We have a sequence of independent, identically distributed RVs X_i, where for each X_i we have:

$$X_i = \begin{cases} 1 & \text{with probability p} \\ 0 & \text{with probability 1-p} \end{cases}$$

For each X_i, we have $E(X_i) = p$ and $\sigma = E(X_i^2) - E^2(X_i) = p(1-p)$. Using the central limit theorem, we have that $Z_n = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is standard normal. Therefore, we want to find n so that

$$P\left(|Z_n| < \frac{10^{-2}\sqrt{n}}{\sigma}\right) \geq 19/20$$

Thus, $\frac{10^{-2}\sqrt{n}}{\sigma} \geq 1.96$ which results in $n \geq 38416 \cdot \sigma^2$. Based on the value of p, the proper value for n can be chosen.

8.

$$\begin{aligned}
R_Y(t_1,t_2) &= E(Y(t_1)Y(t_2))\\
&= E\left(\textstyle\int_{-\infty}^{\infty} h(u_1)X(t_1-u_1)du_1 \int_{-\infty}^{\infty} h(u_2)X(t_2-u_2)du_2\right)\\
&= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)E(X(t_1-u_1)X(t_2-u_2))du_1du_2\\
&= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)R_X(t_1-u_1-t_2+u_2)du_1du_2
\end{aligned}$$