ECSE 305A: Probability and Random Signals I
Problem Set 10
solutions

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1. Note that

$$Var(\alpha X + Y) = \alpha^2 \sigma_X^2 + \sigma_Y^2 + 2\alpha\rho(X,Y)\sigma_X\sigma_Y$$

Setting $\frac{d}{d\alpha} Var(\alpha X + Y) = 0$, we get $\alpha = -\rho(X,Y)\frac{\sigma_Y}{\sigma_X}$.

2. Calculating the $f_X(x)$ and $f_Y(y)$ we can see that the two RV X and Y are independent. Therefore,

$$\begin{aligned} E(X^2Y) &= \int_0^\infty \int_0^\infty 2x^2 y e^{-(x+2y)} dx dy \\ &= \int_0^\infty 2x^2 e^{-x} dx \quad \int_0^\infty y e^{-2y} dy \\ &= 1. \end{aligned}$$

3. The probability density function of Θ is given by

$$f(\theta) = \begin{cases} \frac{1}{2\pi} & \text{if } \theta \in [0, 2\pi] \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$E(XY) = \int_0^{2\pi} \sin\theta \cos\theta \frac{1}{2\pi} d\theta = 0$$

$$E(X) = \int_0^{2\pi} \sin\theta \frac{1}{2\pi} d\theta = 0$$

$$E(Y) = \int_0^{2\pi} \cos\theta \frac{1}{2\pi} d\theta = 0$$

Thus, $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$. They are uncorrelated.

4.

$$\mu_Y = E(Y) = E(X^2) = \sigma^2$$

The covariance between X and Y is:

$$Cov(X,Y) = E(X(Y-\mu_Y)) = E(X^3) - E(X)\mu_Y = 0$$

Since, mean of X is zero and the expected value of X^3 is also zero, because the Gaussian density with zero mean is symmetric, so that all odd moments are zero. You can see that two strongly related random variables have zero covariance and hence are uncorrelated.

5.

$$I = g(R,V) = V/R$$

The average current then becomes

$$\begin{aligned}
E(I) &= E(V/R) \\
&= \int_{-\infty}^{\infty}\int_9^{11} \tfrac{v}{u} f_V(v) f_R(u) du dv \\
&= \left[\int_{-\infty}^{\infty} v f_V(v) dv\right]\left[\int_9^{11} \tfrac{1}{2u} du\right] \\
&= 9\int_9^{11} \tfrac{1}{2u} du \\
&= 0.903
\end{aligned}$$

$$E(I^2) = E(V^2/R^2) = E(V^2)E(1/R^2)$$

Using the fact that R and V are independent.

$$E(V^2) = \mu_Y^2 + \sigma_Y^2 = 81 + 4 = 85$$

$$E(\frac{1}{R^2}) = \int_9^{11} \frac{1}{u^2} 0.5 du = 1/99$$

Hence, $E(I^2) = 85/99 = 0.858586$, and the variance is given by:

$$\sigma_I^2 = E(I^2) - \mu_I^2 = 0.043177$$

6. For $1 \le i \le n$, let X_i be the ith random number selected; we have

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \frac{(1-0)^2}{12} = \frac{n}{12}$$

7. The desired joint marginal probability functions are given by

$$p_{X,Y}(x,y) = \sum_{z=1}^{2} \frac{xyz}{162} = \frac{xy}{54}, \quad x = 4,5, y = 1,2,3$$

$$p_{Y,Z}(y,z) = \sum_{x=4}^{5} \frac{xyz}{162} = \frac{yz}{18}, \quad y = 1,2,3, z = 1,2$$

$$p_{X,Z}(x,z) = \sum_{y=1}^{3} \frac{xyz}{162} = \frac{xz}{27}, \quad x = 4,5, z = 1,2$$

8. They are not independent because $P(X_1 = 1, X_2 = 1, X_3 = 0) = 1/4$, whereas $P(X_1 = 1)P(X_2 = 1)P(X_3 = 0) = 1/8$.

9. $f(x,y,z) = (2x)(2y)(2z)$, $0 < x < 1$, $0 < y < 1$, $0 < z < 1$. Since $2x$, $0 < x < 1$ is a probability density function, $2y$, $0 < y < 1$ is a probability density function, and $2z$, $0 < z < 1$ is also a probability density function, these three function are $f_X(x)$ $f_Y(y)$, and $f_Z(z)$, respective. Therefore, $f(x,y,z) = f_X(x)f_Y(y)f_Z(z)$ showing that X, Y and Z are independent. Thus, $\rho(X,Y) = \rho(Y,Z) = \rho(X,Z) = 0$.