ECSE 305, W09

Assignment # 8, Solutions

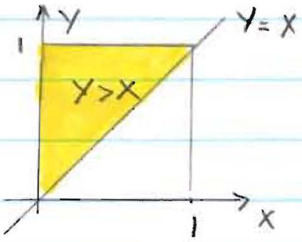
1. a) $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = c\int_0^1\int_0^1 x^k y^l\,dx\,dy$

$$= c\left(\int_0^1 x^k dx\right)\left(\int_0^1 y^l dy\right)$$

$$= \frac{c}{(k+1)(l+1)}$$

$$= 1$$

$$\Longrightarrow \quad c = (k+1)(l+1)$$



b) $P(Y > X) = c\int_0^1 dx \int_x^1 dy\; x^k y^l$

$$= c\int_0^1 dx\, x^k \left(\frac{y^{l+1}}{l+1}\right)_x^1$$

$$= \frac{c}{l+1}\int_0^1 dx\left(x^k - x^{k+l+1}\right)$$

$$= \frac{c}{l+1}\left(\frac{x^{k+1}}{k+1} - \frac{x^{k+l+2}}{k+l+2}\right)_0^1$$

$$= (k+1)\left(\frac{1}{k+1} - \frac{1}{k+l+2}\right)$$

$$= \frac{l+1}{k+l+2}$$

If $l > k$, then $P(Y > X) > 1/2$.

c) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$, $\quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

- $x \notin [0,1] \implies f_X(x) = 0$

  $x \in [0,1] \implies f_X(x) = c x^k \int_0^1 y^l \, dy = (k+1) x^k$

- $y \notin [0,1] \implies f_Y(y) = 0$

  $y \in [0,1] \implies f_Y(y) = c y^l \int_0^1 x^k \, dx = (l+1) y^l$

2. See Figure 1 (next page)

3. The joint PDF of X and Y is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \, e^{-\frac{1}{2}\phi(u,v)}$$

$$u \triangleq \frac{x-\mu_X}{\sigma_X}, \quad v \triangleq \frac{y-\mu_Y}{\sigma_X}$$

$$\phi(u,v) = \frac{1}{1-\rho^2}\left(u^2 - 2\rho uv + v^2\right)$$

We write $\phi(u,v)$ as a perfect square in v plus a residual term in u:

$$\phi(u,v) = \frac{1}{1-\rho^2}\left(v^2 - 2\rho uv + \rho^2 u^2 + u^2 - \rho^2 u^2\right)$$

$$= \frac{1}{1-\rho^2}\left[(v-\rho u)^2 + (1-\rho^2)u^2\right]$$

$$= \frac{(v-\rho u)^2}{1-\rho^2} + u^2$$

Now we have:

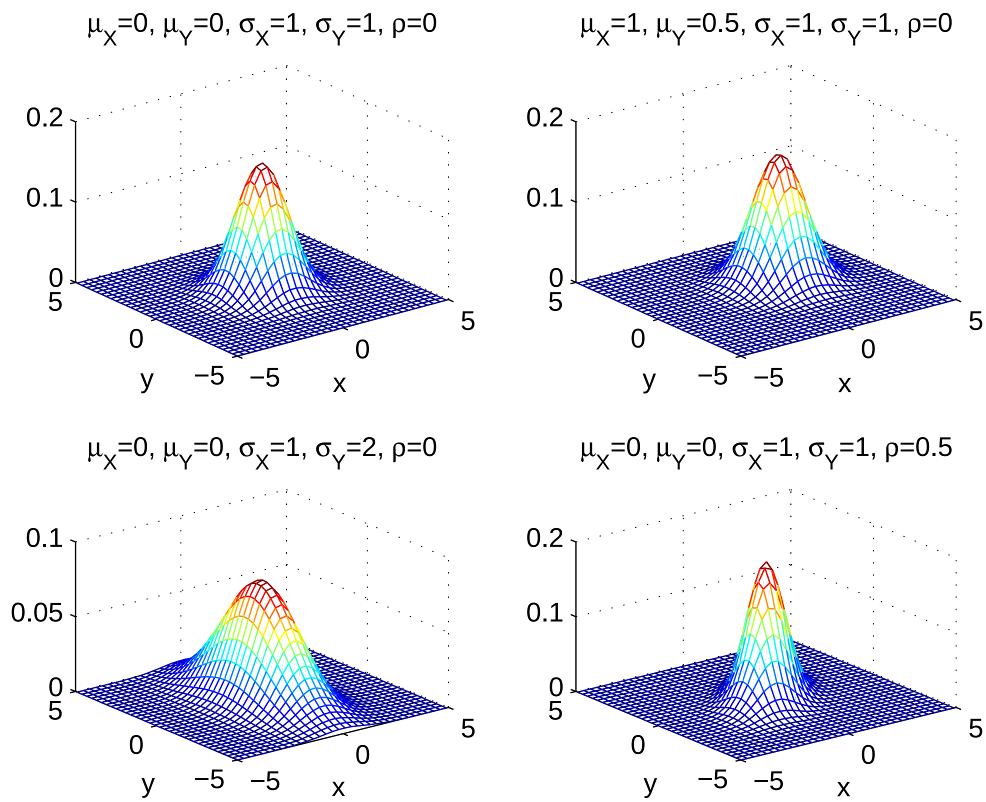


Figure 1: Question 3

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}}\, e^{-\frac{1}{2}u^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(v-\rho u)^2}{(1-\rho^2)}}\, dy$$

To evaluate the integral, we make the change of variable

$$t = \frac{v-\rho u}{\sqrt{1-\rho^2}} = \frac{y-\mu_Y-\rho\sigma_Y u}{\sigma_Y\sqrt{1-\rho^2}}$$

$$dt = \frac{dy}{\sigma_Y\sqrt{1-\rho^2}}$$

Therefore

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X}\, e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \overbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt}^{=\int \text{of } N(0,1) \,=\, 1}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X}\, e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

which shows that $X \sim N(\mu_X, \sigma_X)$. The same approach can be used to show that $Y \sim N(\mu_Y, \sigma_Y)$.

4. The random variables X and Y are not independent (i.e. they are dependent). This can be seen by the following counter-example. We have

$$P(X=8) = \frac{\binom{13}{8}}{\binom{52}{8}} \triangleq p > 0 \qquad (p \simeq 1.71 \times 10^{-6})$$

$$P(Y=8) = \frac{\binom{13}{8}}{\binom{52}{8}} = p$$

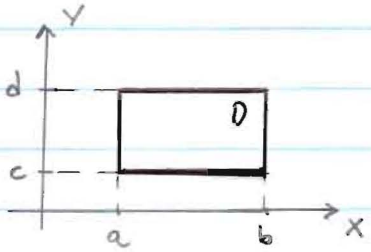
Also, since $X+Y=8$ (with prob. one), we have

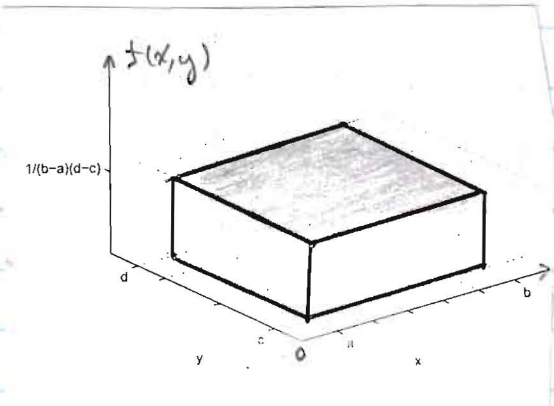
$$P(X=8, Y=8) = 0$$

Since $P(X=8, Y=8) = 0 \neq P(X=8)P(Y=8) = p^2 > 0$, we conclude that X and Y are NOT independent.

5. a)

$$f(x,y) = \begin{cases} \dfrac{1}{(b-a)(d-c)}, & (x,y) \in D \\ 0, & \text{otherwise} \end{cases}$$





b) $F(x,y) = \int_{-\infty}^{x} du \int_{-\infty}^{y} dv \; f(u,v)$

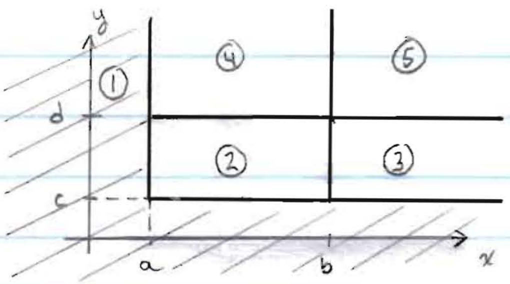
$x < a$ or $y < c \implies F(x,y) = 0$ ①

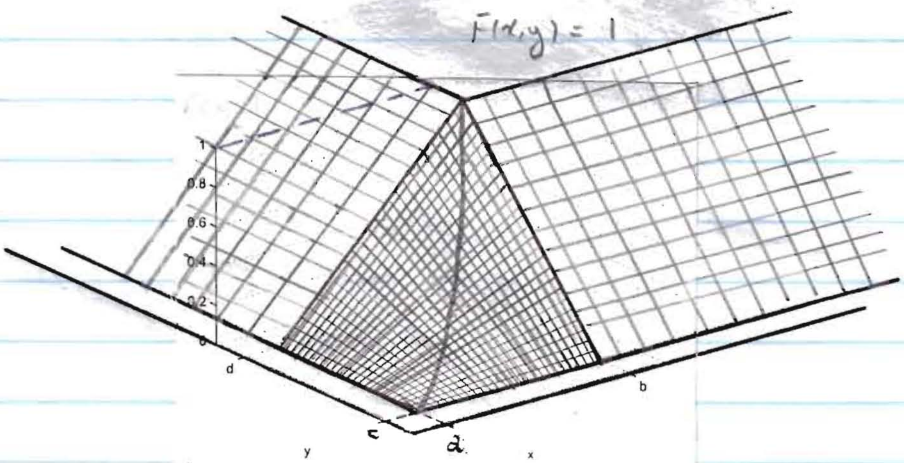
$a \leq x < b$ and $c \leq y < d \implies F(x,y) = \dfrac{(x-a)(y-c)}{(b-a)(d-c)}$ ②

$x \geq b$ and $c \leq y < d \implies F(x,y) = \dfrac{y-c}{d-c}$ ③

$a \leq x < b$ and $y \geq d \implies F(x,y) = \dfrac{x-a}{b-a}$ ④

$x \geq b$ and $y \geq d \implies F(x,y) = 1$ ⑤





c)

$$F_X(x) = F(x,\infty) = \begin{cases} 0 & , \quad x < a \\ \frac{x-a}{b-a} & , \quad a \leq x < b \\ 1 & , \quad x \geq b \end{cases}$$

$$F_Y(y) = F(\infty, y) = \begin{cases} 0 & , \quad y < c \\ \frac{y-c}{d-c} & , \quad c \leq y < d \\ 1 & , \quad y \geq d \end{cases}$$

Using the results in b), we note that $F(x,y) = F_X(x) F_Y(y)$ for all $(x,y) \in \mathbb{R}^2$, so X and Y are independent. In this type of problem, however, it is usually easier to work with PDFs. We have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \frac{1}{b-a} & , \quad a \leq x \leq b \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \begin{cases} \frac{1}{d-c}, & c \le y \le d \\ 0, & \text{otherwise} \end{cases}$$

Looking at the result for $f(x,y)$ in a), we see that $f(x,y) = f_X(x)\, f_Y(y)$ for all $(x,y) \in \mathbb{R}^2$, and so X and Y are independent.

6. The joint PDF of X and Y is

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}$$

Consider the transformation

$$r = \sqrt{x^2+y^2}$$
$$\theta = \measuredangle(x,y)$$

For any $r \in (0,\infty)$ and $\theta \in [0,2\pi)$, this transformation has a unique inverse:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The corresponding Jacobian is

$$J = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

The joint PDF of R and Θ is obtained as

$$g(r,\theta) = f(r\cos\theta, r\sin\theta)\,|J| = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

for $r>0$ and $\theta \in [0,2\pi)$, and $g(r,\theta)=0$ otherwise.

$$g_R(r) = \int_0^{2\pi} g(r,\theta)\, d\theta = \begin{cases} r/\sigma^2\, e^{-r^2/2\sigma^2}, & r>0 \\ 0, & r\leq 0 \end{cases}$$

$$g_\Theta(\theta) = \int_0^{\infty} g(r,\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} dr \qquad (\text{set } u = r^2/2\sigma^2,\ du = r/\sigma^2 dr)$$

$$= \frac{1}{2\pi}\int_0^\infty e^{-u}\, du = \begin{cases} \frac{1}{2\pi}, & 0\leq\theta<2\pi \\ 0, & \text{otherwise} \end{cases}$$

Observe that $g(r,\Theta) = g_R(r)\, g_\Theta(\Theta)$, showing that R and Θ are independent RVs.

7. Observe that $Z = \max(X,Y) \leq z$ iff $X\leq z$ and $Y \leq z$. Thus, invoking the independence of X and Y:

$$\begin{aligned} F_Z(z) &= P(Z\leq z) \\ &= P(X\leq z, Y\leq z) \\ &= P(X\leq z)P(Y\leq z) \\ &= F_X(z)\,F_Y(z) \\ &= (1-e^{-\lambda z})^2\, u(z) \end{aligned}$$

Note: here $F_X(z) = F_Y(z) = (1-e^{-\lambda z})u(z)$ (exp. with λ)

8. Note that (see Th. 11.3)

$$
\begin{aligned}
Var(\alpha X + Y) &= Var(\alpha X) + Var(Y) + 2\, Cov(\alpha X, Y) \\
&= \alpha^2 Var(X) + Var(Y) + 2\alpha\, Cov(X, Y) \\
&= \alpha^2 \sigma_X^2 + \sigma_Y^2 + 2\alpha \rho \sigma_X \sigma_Y
\end{aligned}
$$

Setting $\frac{d}{d\alpha} Var(\alpha X + Y) = 0$, we obtain

$$\alpha = -\rho \frac{\sigma_Y}{\sigma_X}$$

(This corresponds to a minimum since $\frac{d^2}{d\alpha^2} Var(\alpha X + Y) = 2\sigma_X^2 > 0$)

9. $$
\begin{aligned}
E(X^2 Y) &= \iint_{\mathbb{R}^2} x^2 y\, f(x,y)\, dx\, dy \\
&= 2 \int_0^\infty dx\, x^2 e^{-x} \int_0^\infty dy\, y e^{-2y} \\
&= 2 \cdot \frac{2!}{1^3} \cdot \frac{1!}{2^2} = 1
\end{aligned}
$$

where we use the formula $\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$ (n integer, $a > 0$).
You should be able to verify that in this example,
X and Y are indep[ende]nt...

10. The PDF of Θ is

$$f(\theta) = \begin{cases} 1/2\pi, & \theta \in [0, 2\pi) \\ 0, & \text{if not} \end{cases}$$

Therefore

$$E(X) = \int_{-\infty}^{\infty} \sin\theta \, f(\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sin\theta \, d\theta = 0$$

$$E(Y) = \frac{1}{2\pi} \int_0^{2\pi} \cos\theta \, d\theta = 0$$

$$E(XY) = \frac{1}{2\pi} \int_0^{2\pi} \sin\theta \cos\theta \, d\theta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \sin 2\theta \, d\theta = 0$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0$$

This shows that X and Y are uncorrelated.