ECSE 305, W09

Assignment #7, Solutions

1. We can apply relation (8.25) from the class notes:

$$\Psi(\omega) = \sum_{m=0}^{\infty} E(X^m) \frac{(-j\omega)^m}{m!}$$

$$= \sum_{m=0}^{\infty} (m+1)!\, 2^m \frac{(-j\omega)^m}{m!}$$

$$= \sum_{m=0}^{\infty} (m+1)(-2j\omega)^{m+}$$

$$= -\frac{1}{2j}\frac{d}{d\omega}\sum_{m=0}^{\infty}(-2j\omega)^{m+1}$$

$$= -\frac{1}{2j}\frac{d}{d\omega}\left(\frac{1}{1+2j\omega} - 1\right)$$

$$= \frac{1}{(1+2j\omega)^2} \qquad (1)$$

Using a good Handbook of Probability, you can find that (1) is indeed the characteristic function of a gamma RV with parameters $\beta = 2$ and $\lambda = 1/2$. As a check we also note that $\Psi(0) = 1$, which is always the case.

2. a) $$\Psi(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-j\omega x}\, dx$$

$$= \int_0^1 6x(1-x)\, e^{-j\omega x}\, dx$$

$$= 6\frac{e^{-j\omega x}}{-j\omega}\left(x + \frac{1}{j\omega}\right)\Big|_0^1 - 6\frac{e^{-j\omega x}}{-j\omega}\left(x^2 + \frac{2x}{j\omega} + \frac{2}{(j\omega)^2}\right)\Big|_0^1$$

$$= \frac{6}{(j\omega)^2}(e^{-j\omega}+1) + \frac{12}{(j\omega)^3}(e^{-j\omega}-1)$$

b) Substituting the McLaurin series of $e^{-j\omega}$, i.e.
$e^{-j\omega} = 1 - j\omega + (j\omega)^2/2 - (j\omega)^3/6 + (j\omega)^4/24 - (j\omega)^5/120$, we have

$$\psi(\omega) = \frac{6}{(j\omega)^2}\left[\cancel{2} - \cancel{(j\omega)} + (j\omega)^2/2 - (j\omega)^3/6 + (j\omega)^4/24 - \cdots\right]$$

$$+ \frac{12}{(j\omega)^3}\left[-\cancel{(j\omega)} + \cancel{(j\omega)^2/2} - (j\omega)^3/6 + (j\omega)^4/24 - (j\omega)^5/120 + \cdots\right]$$

$$= 1 - \frac{j}{2}\omega - \frac{3}{20}\omega^2$$

$$\psi'(0) = -\frac{j}{2}, \quad \psi''(0) = -\frac{3}{10}$$

$$\mu = E(X) = j\,\psi'(0) = 1/2$$

$$E(X^2) = j^2\psi''(0) = 3/10$$

$$\sigma^2 = E(X^2) - \mu^2 = 1/20$$

3. For a Poisson RV, we have $p(x) = e^{-\lambda}\lambda^x/x!$ with range $R_X = \{0,1,2,\ldots\}$. Therefore

$$\psi(\omega) = \sum_{x=0}^{\infty} e^{-j\omega x}\, p(x)$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{-j\omega})^x}{x!}$$

$$= e^{-\lambda}\, e^{\lambda e^{-j\omega}}$$

$$= e^{\lambda(e^{-j\omega}-1)}$$

Taking derivatives, we find

$$\Psi'(\omega) = -j\lambda e^{-j\omega} e^{\lambda(e^{-j\omega}-1)}$$

$$\Psi''(\omega) = \left(-\lambda^2 e^{-2j\omega} - \lambda e^{-j\omega}\right) e^{\lambda(e^{-j\omega}-1)}$$

$$\mu = E(X) = j\Psi'(0) = j(-j\lambda) = \lambda$$

$$E(X^2) = j^2\Psi''(0) = \lambda^2 + \lambda$$

$$\sigma^2 = E(X^2) - \mu^2 = \lambda$$

4. You could find the PDF of Y, say $f_Y(y)$, and then compute $\Psi_Y(\omega) = \int f_Y(y) e^{-j\omega y} dy$. However it is simpler to proceed as follows:

$$\Psi_Y(\omega) = E[e^{j\omega Y}]$$

$$= E[e^{j\omega(2X+1)}]$$

$$= e^{j\omega} E[e^{j2\omega X}]$$

$$= e^{j\omega}\Psi_X(2\omega) = \frac{e^{j\omega}}{1+2j\omega}$$

5. a) $\sum_{x=1}^{3}\sum_{y=1}^{3} p(x,y) = k\sum_{i=1}^{3}\sum_{j=1}^{3}(i+j)$

$$= 2k\sum_{i=1}^{3}\sum_{j=1}^{3} i \qquad \text{(due to symmetry)}$$

$$= 6k\sum_{i=1}^{3} i = 36k$$

This sum must be equal to 1. Therefore $k = 1/36$

b) $P(X=1, Y<3) = p(1,1) + p(1,2)$
$= 2/36 + 3/36 = 5/36$

$P(X=1, Y\leq 3) = p(1,1) + p(1,2) + p(1,3) = 9/36 = 1/4$

$P(X=2) = p(2,1) + p(2,2) + p(2,3) = 12/36 = 1/3$

$P(X<Y) = p(1,2) + p(1,3) + p(2,3) = 12/36 = 1/3$

$P(X\leq Y) = P(X<Y) + P(X=Y)$
$= 12/36 + 12/36 = 2/3$

6. a) $\sum_{\text{all}(i,j)} p(i,j) = k(2+10+13)$
$= 25k$
$= 1 \implies k = 1/25$

b) Let $R_X = \{1,2\}$ and $R_Y = \{1,3\}$.

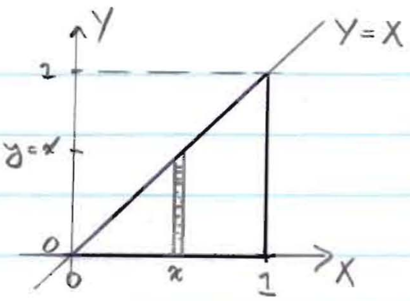
- $p_X(x) = \sum_{y\in R_Y} p(x,y) = p(x,1) + p(x,3)$

$$\left.\begin{array}{l} x=1 \rightarrow p_X(x) = 2/25 + 10/25 = 12/25 \\ x=2 \implies p_X(x) = 0 + 13/25 = 13/25 \end{array}\right\} \text{Note: } \Sigma = 1$$

- $p_Y(y) = \sum_{x\in R_X} p(x,y) = p(1,y) + p(2,y)$

$y=1 \implies p_Y(y) = 2/25 + 0 = 2/25$
$y=3 \implies p_Y(y) = 10/25 + 13/25 = 23/25$

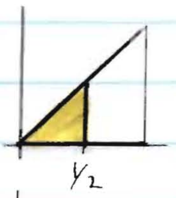
7 a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$



$x \in [0,1] \Longrightarrow f_X(x) = \int_0^x 2\,dy = 2x$
$x \notin [0,1] \Longrightarrow f_X(x) = 0$

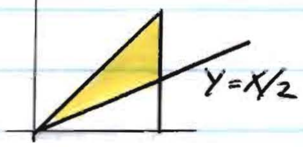
$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$y \in [0,1] \Longrightarrow f_Y(y) = \int_y^1 2\,dx = 2(1-y)$
$y \notin [0,1] \Longrightarrow f_Y(y) = 0$



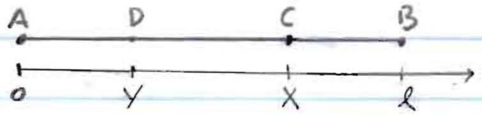
b) $P(X<1/2) = \int_0^{1/2} dx \int_0^x dy\, 2 = 1/4$

$P(X<2Y) = \int_0^1 dx \int_{x/2}^x dy\, 2 = 1/2$



$P(X=Y) = 0$

8. The problem is equivalent to the following: Two random numbers X and Y are selected at random and independently from $[0,\ell]$. What is the probability that $|X-Y| < X$:



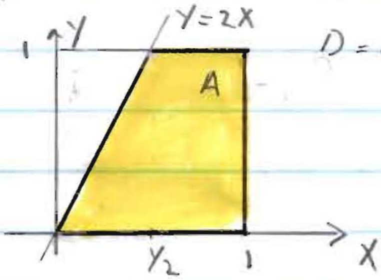
The joint PDF of X and Y is obtained as follows:

$$f_X(x) = \begin{cases} 1/\ell, & x \in [0,1] \\ 0, & \text{if not} \end{cases}$$

$$f_Y(y) = \begin{cases} 1/\ell, & y \in [0,1] \\ 0, & \text{if not} \end{cases}$$

$$f(x,y) = f_X(x)\, f_Y(y) = \begin{cases} 1/\ell^2, & (x,y) \in [0,1]^2 = D \\ 0, & \text{if not} \end{cases}$$

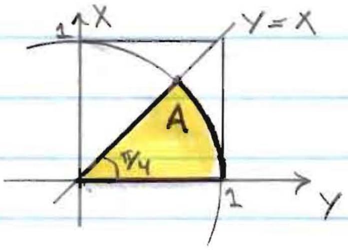
so that X an Y are jointly uniform over the unit square D. We seek $P(A)$, where $A = \{(x,y) \in D : |x-y| < x\}$ is illustrated below:



$D = [0,1] \times [0,1]$ (the square)

$$P(A) = \frac{\text{Area}(A)}{\text{Area}(D)} = \frac{3}{4}$$

9. Let $D = [0,1]^2$ and $A = \{(x,y) \in D : y \leq x \text{ and } x^2 + y^2 \leq 1\}$, as illustrated below:



Note: the coordinates of $A = (a,b)$ are $a = \cos\frac{\pi}{4} = 1/\sqrt{2}$

$b = \sin\frac{\pi}{4} = 1/\sqrt{2}$

$$P(A) = \frac{\text{Area}(A)}{\text{Area}(D)} = \frac{\pi}{8}$$