ECSE 305, W09

Assignment #6, Solutions

1. a) $E(X) \triangleq \sum_{x=1}^{5} x\, p(x) = \frac{1}{15} \sum_{x=1}^{5} x^2 = \frac{55}{15} = \frac{11}{3}$

$E(X^2) = \sum_{x=1}^{5} x^2 p(x) = \frac{1}{15} \sum_{x=1}^{5} x^3 = \frac{225}{15} = 15$

$Var(X) = E(X^2) - E(X)^2 = 15 - \frac{121}{9} = \frac{14}{9}$

b) From the table, we see that $R_Y = \{5, 8, 9\}$.
If $y \notin R_Y$, then $p_Y(y) = 0$

X	Y
1	5
2	8
3	9
4	8
5	5

$p_Y(5) = P(Y=5)$
$= P(X=1) + P(X=5)$
$= 1/15 + 5/15 = 2/5$

$p_Y(8) = P(X=2) + P(X=4)$
$= 2/15 + 4/15 = 2/5$

$p_Y(9) = P(X=3) = 1/5$

c) $E(Y) = \sum_{y \in R_Y} y\, p_Y(y) = 5 \cdot \frac{2}{5} + 8 \cdot \frac{2}{5} + 9 \cdot \frac{1}{5} = 7$

$E(Y) = E(X(6-X)) = E(6X - X^2)$
$= 6E(X) - E(X^2) = 6 \cdot 11/3 - 15 = 7$

$E(Y) = \sum_{x=1}^{5} x(6-x)\, p(x) = \ldots = 7$

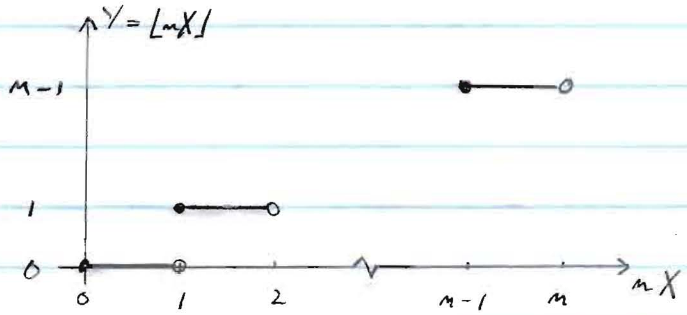
2. Let X denote the number of errors on the specific page. Then X is a Poisson random variable with $E(X) = \lambda = 1/5$. We want

$$
\begin{aligned}
P(X \geq 1) &= 1 - P(X=0) \\
&= 1 - e^{-\lambda} \\
&\simeq 0.18
\end{aligned}
$$

3. X is a continuous RV with uniform PDF

$$
f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}
$$

For m a positive integer, define $Y = \lfloor mX \rfloor$. The situation is illustrated below:



We see that Y is a discrete RV from the set $R_Y = \{0, 1, \ldots, m-1\}$. For any $y \in R_Y$, we have

$$
\begin{aligned}
p_Y(y) &= P(Y = y) \\
&= P(y \leq mX < y+1) \\
&= P\left(\tfrac{y}{m} \leq X < \tfrac{y+1}{m}\right) = 1/m
\end{aligned}
$$

4. The length of the other side is $Y = \sqrt{81 - X^2}$. Therefore, the expected value is

$$E[Y] = \int_{-\infty}^{\infty} \sqrt{81-x^2}\, f(x)\, dx$$

$$= \frac{1}{6} \int_{2}^{4} \sqrt{81-x^2}\, x\, dx$$

$$= \frac{1}{12} \int_{65}^{77} \sqrt{u}\, du \, , \qquad u = 81 - x^2, \quad du = -2x\, dx$$

$$= \frac{1}{18} u^{3/2} \Big|_{65}^{77}$$

$$\simeq 8.4$$

5. More generally, consider the Laplacian RV (Section 7.5.3) with PDF

$$f(x) = \frac{\alpha}{2} e^{-\alpha|x|} \, , \qquad x \in \mathbb{R}$$

where $\alpha > 0$ is an arbitrary parameter. Note that

$$\int_{-\infty}^{\infty} f(x)\, dx = \alpha \int_{0}^{\infty} e^{-\alpha x}\, dx = 1 .$$

Also, by symmetry, $E(X) = \mu = 0$. We have

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \alpha \int_{0}^{\infty} x^2 e^{-\alpha x}\, dx$$

$$\overset{(1)}{=} \alpha \frac{e^{-\alpha x}}{(-\alpha)} \left(x^2 + \frac{2x}{\alpha} + \frac{2}{\alpha^2} \right) \Big|_{0}^{\infty}$$

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(1) This is from Table of integrals; can also do integration by parts

$$= \frac{2}{\alpha^2}$$

$$Var(X) = E(X^2) - \cancel{\mu^2} = \frac{2}{\alpha^2}$$

In the special case of interest here, i.e. $\alpha = 1$, we find $Var(X) = 2$.

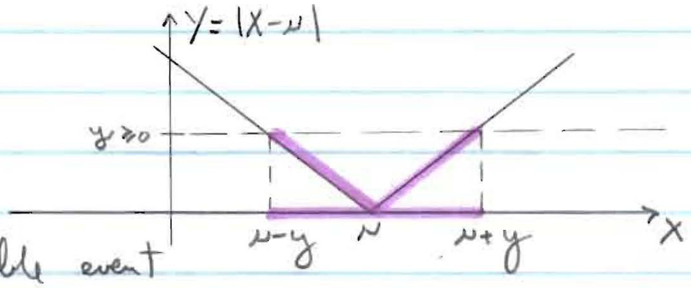
6. $$E[e^X] = \int_{-\infty}^{\infty} e^x f(x)\, dx$$

$$= 3\int_0^{\infty} e^{-2x}\, dx$$

$$= 3\left(\frac{e^{-2x}}{-2}\right)\Bigg|_0^{\infty} = \frac{3}{2}$$

7. Different approaches are possible for finding the CDF of $Y = |X-\mu|$.

A) <u>Method of distribution</u>:



Step ①:

case $y<0$: $\quad Y \leq y \iff$ impossible event

case $y \geq 0$: $\quad Y \leq y \iff \mu - y \leq X \leq \mu + y$

Step ②:

case $y<0$: $\quad G(y) = P(Y \leq y) = 0$

case $y \geq 0$: $\quad G(y) = P(\mu - y \leq X \leq \mu + y)$

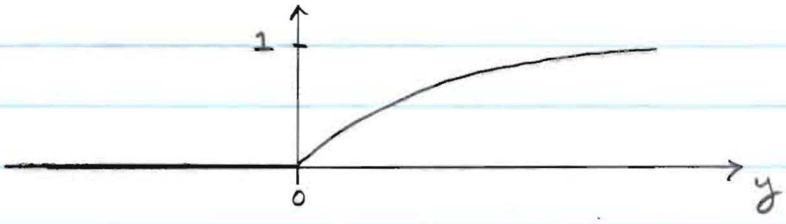
$$= P(X \leq \mu + y) - P(X < \mu - y)$$

$$= \Phi\left(\frac{y}{\sigma}\right) - \Phi\left(-\frac{y}{\sigma}\right) \qquad \left(\text{Note: } \Phi(-x) = 1 - \Phi(x)\right)$$

$$= 2\Phi\left(\frac{y}{\sigma}\right) - 1$$

Therefore, the CDF of Y is

$$G(y) = \begin{cases} 0, & y \leq 0 \\ 2\Phi(\frac{y}{\sigma}) - 1, & y > 0 \end{cases}$$



B) <u>Method of transformation</u>: For $y < 0$, $y = |x - \mu|$ has no root and so $g(y) = 0$. For $y > 0$, there are 2 roots:

$$x_1 = \mu - y \quad , \quad dx_1/dy = -1$$
$$x_2 = \mu + y \quad , \quad dx_2/dy = +1$$

$$g(y) = \sum_{i=1}^{2} f(x_i) \left| \frac{dx_i}{dy} \right|$$

$$= f(\mu - y) + f(\mu + y) \quad , \quad f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\phi\left(-\frac{y}{\sigma}\right) + \frac{1}{\sigma}\phi\left(\frac{y}{\sigma}\right) , \quad y > 0$$

Finally, $G(y)$ is obtained by integration:

$$G(y) = \int_{-\infty}^{y} g(t)\, dt$$

$$= \begin{cases} 0 & y < 0 \\ \Phi(\frac{y}{\sigma}) - \Phi(-\frac{y}{\sigma}) & y \geq 0 \end{cases}$$

For the expected value, we have

$$E[Y] = E[|X-\mu|] = \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

We make the change of variable $t \triangleq (x-\mu)/\sigma$, $dt = dx/\sigma$:

$$E[Y] = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} t\, e^{-t^2/2}\, dt$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma\, e^{-t^2/2} \Big|_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma$$

8. Let X denote the chest size of a randomly selected soldier. We assume that $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 40$ in and $\sigma = 2.5$ (the inflection pts are located at $\mu \pm \sigma$ for a normal distribution). Then

$$p \triangleq P(X \geq 40)$$

$$= P\left(\frac{X-40}{2.5} \geq \frac{40-40}{2.5}\right)$$

$$= P(Z \geq 0) = \frac{1}{2} \quad \text{where } Z \triangleq \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

Now, let Y denote the number of soldiers, out of 50, with chest size of at least 40. We model $Y \sim \mathcal{B}(n, p)$ with $n = 50$ and $p = 1/2$. We seek

$$P(Y = 10) = \binom{50}{10} (.5)^{10} (.5)^{40} \simeq 9.12 \times 10^{-6}$$

9\. a) Let X denote the score of a randomly selected student. Then $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 75$ and $\sigma = 10$. We seek α such that $P(X \geqslant \alpha) = .1$. Equivalently:

$$P(X \leqslant \alpha) = .9 \iff P\left(\frac{X-\mu}{\sigma} < \frac{\alpha-\mu}{\sigma}\right) = .9$$

$$\iff \Phi\left(\frac{\alpha-\mu}{\sigma}\right) = .9$$

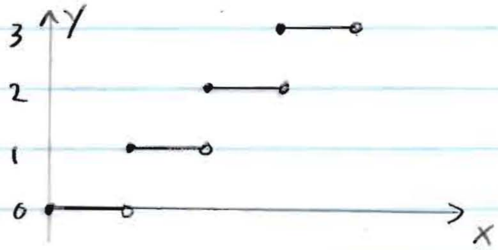
since $Z \triangleq \frac{X-\mu}{\sigma}$ is $\mathcal{N}(0,1)$. From the table for $\Phi(\cdot)$, we have that $\Phi(1.28) \cong .8997$. Then

$$\frac{\alpha-\mu}{\sigma} = 1.28$$

$$\begin{aligned}\alpha &= \mu + 1.28\sigma \\ &= 75 + 12.8 \\ &= 87.8\ \%\end{aligned}$$

b) We just have to use different values from the table:

5% : $\Phi(1.59) \simeq .9441 \longrightarrow \alpha = 90.9\%$
2% : $\Phi(2.06) \simeq .9803 \longrightarrow \alpha = 95.6\%$

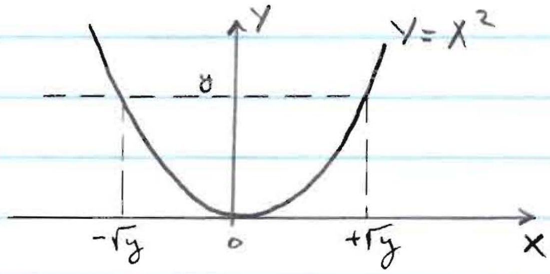


10\. We note that Y can only take integer values. Also, for negative integers y, $P(Y = y) = 0$. For non-negative integer, i.e. $y \in Q_y = \{0, 1, 2, \ldots\}$, we have

$$p(y) \triangleq P(Y = y) = P(y \leqslant X < y+1)$$

$$= \int_y^{y+1} \lambda e^{-\lambda x}\, dx = e^{-\lambda y}(1-e^{-\lambda})$$

Introducing $p = 1-e^{-\lambda}$, we have $p(y) = q^y p$ which corresponds to the PMF of a geometric RV.



11. We can use the method of distribution with $Y = X^2$. For $y < 0$, we have

$$G(y) = P(Y \leq y)$$
$$= P(X^2 \leq y) = 0$$

$$g(y) = G'(y) = 0$$

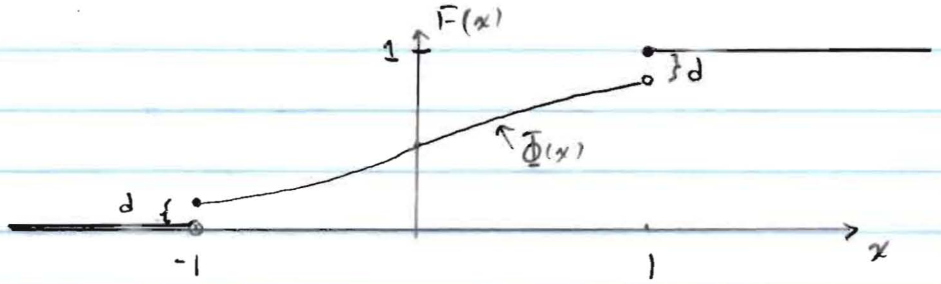
For $y \geq 0$, we have $Y \leq y \iff -\sqrt{y} \leq X \leq \sqrt{y}$. So

$$G(y) = P(Y \leq y)$$
$$= P(-\sqrt{y} \leq X \leq \sqrt{y})$$
$$= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y})$$
$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$
$$= 2\,\Phi(\sqrt{y}) - 1$$

$$g(y) = \frac{1}{\sqrt{y}} \phi(\sqrt{y}) = \frac{1}{\sqrt{y}\sqrt{2\pi}} e^{-y/2}$$
$$= \frac{1}{\sqrt{\pi}} \frac{1}{2} e^{-y/2} (y/2)^{-1/2}$$
$$= \frac{\lambda}{\Gamma(\beta)} e^{-\lambda y} (\lambda y)^{\beta - 1}$$

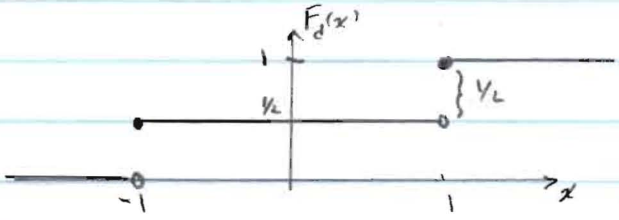
with $\lambda = \frac{1}{2}$, $\beta = 1/2$ and $\Gamma(1/2) = \sqrt{\pi}$ (see Table of gamma function)

12.

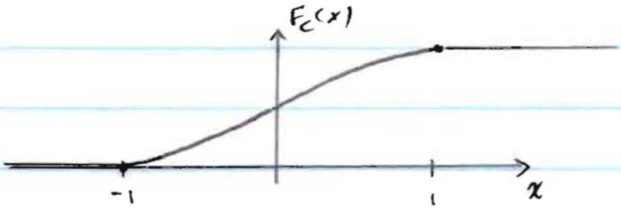


a) It is convenient to define $d \triangleq \Phi(-1) = 1 - \Phi(1) \simeq 0.1587$.
Clearly, we can write $F(x) = \alpha F_d(x) + \beta F_c(x)$ where we define

$$F_d(x) \triangleq \begin{cases} 0, & x < -1 \\ 1/2, & -1 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$



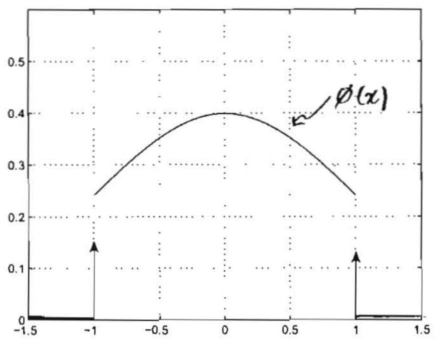
$$F_c(x) \triangleq \begin{cases} 0, & x < -1 \\ \dfrac{\Phi(x) - d}{1 - 2d}, & -1 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$



Parameter α is obtained by adjusting the size of the jump at $x = -1$: $\frac{\alpha}{2} = d$, i.e. $\alpha = 2d = .3174$ and $\beta = 1 - \alpha = .6826$

b) From the plot of $F(x)$ at the top:

$$f(x) = \frac{d}{dx} F(x) = d\,\delta(x+1) + d\,\delta(x-1) + \phi(x)\,[u(x+1) - u(x-1)]$$



c) $P(X<0) = F(0^-) = \Phi(0^-) = \Phi(0) = 1/2$

$P(X \leq 0) = F(0) = \Phi(0) = 1/2$

$P(X<1) = F(1^-) = \Phi(1^-) = \Phi(1) = .8413$

$P(X \leq 1) = F(1) = 1$

d) $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= (-1)d + (1)d + \int_{-1}^{1} x \phi(x)\, dx$

$= 0$

$Var(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= 2d + \int_{-1}^{1} x^2 \phi(x)\, dx$

$= \int_{-\infty}^{-1} \phi(x)\, dx + \int_{1}^{\infty} \phi(x)\, dx + \int_{-1}^{1} x^2 \phi(x)\, dx$

$< \int_{-\infty}^{\infty} x^2 \phi(x)\, dx = 1$