1, a) Let $X \triangleq$ number of shots until 1st basket. Then X is geometric with parameter $p = 0.45$ $(q = 0.55)$ and its PMF is

$$p(k) \triangleq P(X=k) = q^{k-1}p \quad , \quad k \in R_X = \{1, 2, 3, \ldots$$

The desired probability is

$$P(X>6) = 1 - P(X \leq 6)$$

$$= 1 - \sum_{k=1}^{6} q^{k-1} p = 0.0027$$

b) Let $Y \triangleq$ number of shots, after 1st basket, until 2nd basket. Because we "reset" counting, Y is also geometric with $p = .45$ and X and Y are independent. We seek

$$P(X=4 \text{ and } Y<4) = P(X=4)\,P(Y<4)$$

$$= q^3 p \left(\sum_{k=1}^{3} q^{k-1} p\right) = 0.0624$$

2. Let X denote the number of persons who decide correctly among a 3-person jury. Then $X \sim B(3,p)$. The probability that a 3-person jury decides correctly is

$$P(X \geq 2) = P(X=2) + P(X=3)$$

$$= \binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3$$

$$= 3p^2 - 2p^3$$

A 3-person jury is preferable to a single juror iff the prob. of making the correct decision is increased, i.e.

$$3p^2 - 2p^3 > p$$

$$\Longleftrightarrow \quad 3p - 2p^2 - 1 > 0 \quad , \quad (\text{assume } 0 < p < 1)$$

$$'' \quad 2(1-p)(p - 1/2) > 0$$

$$'' \quad p > 1/2$$

Finally, a 3-person jury is preferable if $p > 1/2$. In case $p < 1/2$, a single juror is preferable and if $p = 1/2$, there is no difference.

3. a) Let X be the number of members born on January 1^{st}. Then $X \sim \mathcal{B}(n, p)$ with parameters $n = 45$ and $p = 1/365$. Therefore

$$p_i = P(X=i) = \binom{n}{i} p^i q^{n-i}, \qquad q = 1-p$$

$$= \binom{45}{i}\left(\frac{1}{365}\right)^i \left(\frac{364}{365}\right)^{n-i}$$

$$p_0 \simeq 0.884$$
$$p_1 \simeq 0.109$$
$$p_2 \simeq 0.0066$$
$$p_3 \simeq 0.00026$$

b) We can approximate X as a Poisson RV with parameter $\lambda = np = 45 \cdot \frac{1}{365} \simeq 0.12$. Then

$$p_i = P(X=i) \simeq e^{-\lambda}\frac{\lambda^i}{i!}$$

$$= e^{-0.12}\frac{(0.12)^i}{i!}$$

$$p_0 \simeq 0.884$$
$$p_1 \simeq 0.109$$
$$p_2 \simeq 0.0067$$
$$p_3 \simeq 0.00027$$

4. Let ρ denote the probability of reaching a decision on a certain round of coin tossing:

$$\rho = P(\text{2H and 1T OR 1H and 2T})$$

$$= \binom{3}{2}p^2 q + \binom{3}{1} p q^2$$

$$= 3pq(p+q) = 3pq$$

The prob. of not reaching a decision is $r \triangleq 1-p = 1-3pq$.

a) Let X denote the number of tosses until they reach the decision. Then X is geometric with parameter $p = 3pq$. We seek

$$P(X < m) = 1 - P(X \geq m)$$
$$= 1 - r^{m-1}$$
$$= 1 - (1-3pq)^{m-1}$$

$$P(X \geq m) = \sum_{k=m}^{\infty} p r^{k-1} = p r^{m-1} \sum_{k=0}^{\infty} r^k = p r^{m-1} \cdot \frac{1}{1-r} = r^{m-1}$$

b) We want to find the minimum integer m such that $P(X \leq m) \geq 0.95$, in the special case when $p = q = \frac{1}{2}$. This is equivalent to $P(X > m) \leq 0.05$.

$$P(X > m) = r^m$$
$$= (1-3pq)^m$$
$$= (1 - 3/4)^m = 1/4^m$$

Thus, $(1/4)^m \leq .05 \Rightarrow m \geq 2.16$. The smallest m is 3.

5. a) $R_X = \{r, r+1, r+2, \ldots\}$

b) For any $k \in R_X$, we have $X = k$ iff there are $r-1$ successes in the first $k-1$ trials and the last trial is a success. Thus, because the trials are independent:

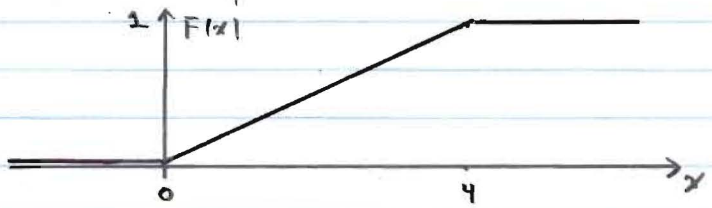
$$p(k) = P(X = k)$$
$$= \binom{k-1}{r-1} p^{r-1} q^{k-1-(r-1)} \cdot p$$
$$= \binom{k-1}{r-1} p^r q^{k-r}, \quad k \in R_X \text{ and } q \triangleq 1-p$$

c) Let X denote the number of games until Bill wins $r=5$ games. Also let Y be the number of games until Monica wins 5 games. Then X and Y are neg. binomial with parameters (r, p) and $(r, 1-p)$, respectively, where $p = 0.42$. We want

$$P(X=8) + P(Y=8) = \binom{7}{4} 0.42^5 \cdot 0.58^3 + \binom{7}{4} 0.58^5 \cdot 0.42^3$$

$$\simeq 0.2594$$

6\. a) The CDF of X is sketched below:



X is a cont. RV because F(X) is absolutely continuous, that is:

- $F(x)$ is continuous for all $x \in \mathbb{R}$
- $F'(x)$ exists everywhere, except for $x \in \{0, 4\}$ (a finite set of points)

b) The PDF is obtained as the derivative of $F(x)$:

$$f(x) = F'(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

c) Here, it is easy to express all the desired probabilities in terms of $F(x)$:

$$P(X \geq 5) = 1 - P(X < 5) = 1 - F(5^-) = 1 - 1 = 0$$
$$P(X < 0) = F(0^-) = 0$$
$$P(X \leq 0) = F(0) = 0$$
$$P(\tfrac{1}{4} \leq X < 1) = F(1^-) - F(1/4^-) = 1/4 - 1/16 = 3/16$$
$$P(1/4 \leq X \leq 1) = F(1^-) - F(1/4) = 1/4 - 1/16 = 3/16$$
$$P(X > 2) = 1 - P(X \leq 1/2) = 1 - F(1/2) = 1 - 1/8 = 7/8$$

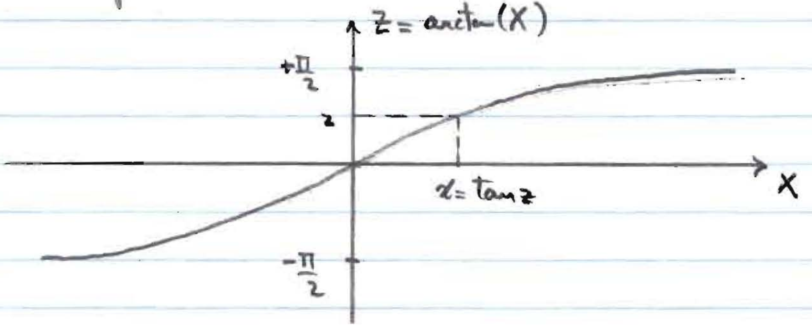
8\. The prob. that a randomly selected radio last more than 15 years is

$$P(X > 15) = \int_{15}^{\infty} f(x)\,dx = \frac{1}{15}\int_{15}^{\infty} e^{-x/15}\,dx = \frac{1}{e}$$

Out of $n = 8$ such radios (i.e. idependent and identical conditions of selection), let Y denote the number of radios that last more than 15 years. Then Y is binomial with parameters $n = 8$ and $p = 1/e$. We seek

$$P(Y \geq 4) = \sum_{k=4}^{8} \binom{8}{k} \left(\frac{1}{e}\right)^k \left(1 - \frac{1}{e}\right)^{8-k} = 0.3327$$

8. The transformation $Z = \arctan(X)$ is sketched below.



A) Method of distribution:

Step ①: case $z \geq \frac{\pi}{2}$: $Z \leq z \iff X \in \mathbb{R}$ = certain event

$-\frac{\pi}{2} < z < \frac{\pi}{2}$: $Z \leq z \iff X \leq x = \tan z$

$z \leq -\frac{\pi}{2}$: $Z \leq z$ = impossible event

Step ②: The CDF of Z is $G(z) \triangleq P(Z \leq z)$

$z \geq \frac{\pi}{2}$: $G(z) = P(S) = 1$

$-\frac{\pi}{2} < z < \frac{\pi}{2}$: $G(z) = P(X \leq \tan z)$

$$= \int_{-\infty}^{\tan z} \frac{1}{\pi(1+x^2)} \, dx$$

$$= \frac{1}{\pi} \arctan x \Big|_{-\infty}^{\tan z}$$

$$= \frac{1}{\pi}\left(z + \frac{\pi}{2}\right)$$

$$= \frac{z}{\pi} + \frac{1}{2}$$

Step ③ : The PDF of Z is $g(z) = G'(z)$:

$$g(z) = \begin{cases} 0, & z < -\pi/2 \\ 1/\pi, & -\pi/2 < z < \pi/2 \\ 0, & z > \pi/2 \end{cases}$$

B) Method of transformation (i.e. using theorem):

Case $|z| \geq \pi/2$: $z = \arctan x$ has no root and so $g(z) = 0$

Case $-\pi/2 < z < \pi/2$: $z = \arctan x$ has a single root $x = \tan z$

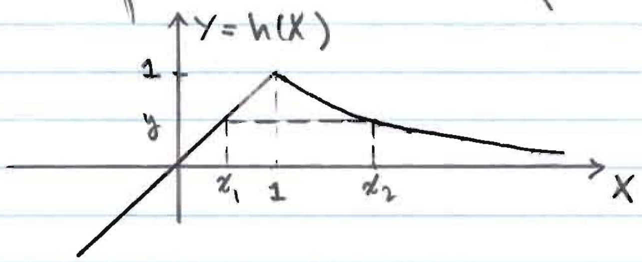
$$\frac{dx}{dz} = \sec^2 z = 1 + \tan^2 z = 1 + x^2$$

$$g(z) = \sum_{\text{all } i} f(x_i) \left| \frac{dx_i}{dz} \right| \quad (\Sigma \text{ over all roots})$$

$$= \frac{1}{\pi(1+x^2)} \left| \frac{dx}{dz} \right| \quad (x = \tan z)$$

$$= \frac{1}{\pi}$$

9. The transformation $Y = h(X)$ of interest is illustrated below:



We use the method of transformation:

- case $y < 0$: $y = h(x)$ has a single root $x = y$ with $\frac{dx}{dy} = 1$

$$g(y) = f(x) = 0 \quad (\text{because } x = y < 0)$$

- $0 < y < 1$: $y = h(x)$ has 2 roots :

$$x_1 = y, \quad \frac{dx_1}{dy} = 1$$

$$x_2 = \frac{1}{y}, \quad \frac{dx_2}{dy} = -\frac{1}{y^2}$$

$$g(y) = f(x_1)\left|\frac{dx_1}{dy}\right| + f(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= f(y) + \frac{1}{y^2} f\left(\frac{1}{y}\right)$$

$$= e^{-y} + \frac{1}{y^2} e^{-1/y}$$

- $y > 1$: $y = h(x)$ has no root so $g(y) = 0$

Finally :

$$g(y) = \begin{cases} e^{-y} + 1/y^2 \, e^{-1/y}, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

The values of $g(y)$ at $y = 0$ and $y = 1$ are of no importance here because $P(Y=0) = P(Y=1) = 0$:

$$P(Y=0) = P(X=0) = 0 \quad (\text{for the given PDF})$$

$$P(Y=1) = P(X=1) = 0$$