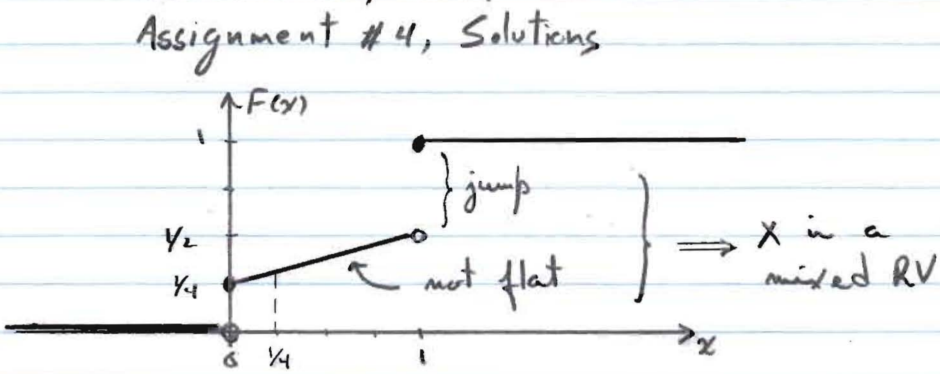
# ECSE 305, W09
# Assignment #4, Solutions

1. (a)




(b) $P(X \geq 5) = 1 - P(X < 5) = 1 - F(5^-) = 0$

$P(X < 0) = F(0^-) = 0$

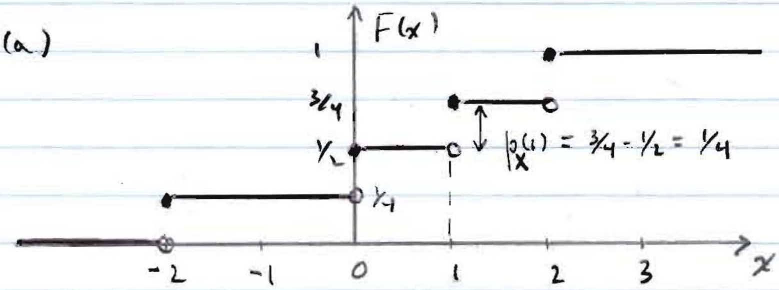
$P(X \leq 0) = F(0) = 1/4$

$P(1/4 \leq X < 1) = F(1^-) - F(1/4^-) = 1/2 - 5/16 = 3/16$

$P(1/4 \leq X \leq 1) = F(1) - F(1/4^-) = 1 - 5/16 = 11/16$

$P(X > 1/2) = 1 - F(1/2) = 5/8$

2. (a)




It can be seen that $F(x)$ is "flat", i.e. $F'(x) = 0$, except for a finite number of jumps. Therefore, X is a discrete RV with $R_X = \{-2, 0, 1, 2\}$.

(b) By definition, the PMF of X is

$$p_X(x) \triangleq P(X = x) = F(x) - F(x^-)$$

If $x \notin R_X$, then $p(x) = 0$, while for $x \in R_X$:

$$p_X(-2) = F(2) - F(2^-) = 1/4 - 0 = 1/4$$
$$p_X(0) = F(0) - F(0^-) = 1/2 - 1/4 = 1/4$$
$$p_X(1) = 1/4$$
$$p_X(2) = 1/4$$

(c) Let $Y = X^2$. The possible values (i.e. with non-zero prob.) of $Y$ are $R_Y = \{0, 1, 4\}$. The PMF of $Y$ is

$$p_Y(y) \triangleq P(Y=y) = P(X^2 = y) = \sum_{x^2=y} p_X(x)$$

$$p_Y(0) = p_X(0) = 1/4$$
$$p_Y(1) = p_X(1) = 1/4$$
$$p_Y(4) = p_X(2) + p_X(-2) = 1/2$$

3. We note that $X$ is a geometric RV with parameter $p = 1/6$, i.e. the probability of a 6 in any given roll. The range of $X$ is $R_X = \{1, 2, 3, \ldots\}$ (positive integers) and the PMF is

$$p(i) = \left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right), \quad i = 1, 2, 3, \ldots$$

The CDF $F(x)$ is a staircase function with jumps at the point of $R_X$:

- For $x < 1$, we have $F(x) = P(X \leq x) = 0$
- If $x \geq 1$, then for some positive integer $m$, $m \leq x < m+1$, and we have that

$$F(x) = P(X \leq x) = \sum_{i=1}^{m} p(i) = \frac{1}{6}\sum_{i=1}^{m}\left(\frac{5}{6}\right)^{i-1} = \frac{1}{6}\,\frac{1-\left(\frac{5}{6}\right)^m}{1-\frac{5}{6}} = 1 - \left(\frac{5}{6}\right)^m$$

Hence

$$F(x) = \begin{cases} 0, & x < 1 \\ 1 - \left(\frac{5}{6}\right)^m, & m \leq x < m+1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

4. $\sum_{k=0}^{\infty} P(X>k) = P(X>0)$
$+P(X>1)$
$+P(X>2) + \ldots$

$$= P(X=1) + P(X=2) + P(X=3) + \ldots$$
$$+ P(X=2) + P(X=3) + \ldots$$
$$+ P(X=3) + \ldots$$

$$= 1\cdot p(1) + 2\cdot p(2) + 3\,p(3) + \ldots$$

$$= \sum_{k=0}^{\infty} k\,p(k)$$

$$= E(X)$$

5. We have $Y = h(X)$ where $h(x) \triangleq (x-1)\,u(x-1)$. Therefore

$$E(Y) = \sum_i h(x_i)\,p_X(x_i) \qquad (\text{Th. } 6.4)$$

$$= \sum_{k=0}^{\infty} (k-1)\,u(k-1)\,p_k \quad , \quad u(k-1) = \begin{cases} 1, & k \in \{1,2,\ldots\} \\ 0, & \text{if not} \end{cases}$$

$$= \sum_{k=1}^{\infty} (k-1)\,p_k$$

$$= \sum_{k=1}^{\infty} k\,p_k - \sum_{k=1}^{\infty} p_k$$

$$= \left(\sum_{k=0}^{\infty} k\,p_k - 0\right) - \left(\sum_{k=0}^{\infty} p_k - p_0\right)$$

$$= E(X) - 1 + p_0$$

Proceeding in the same way:

$$E(Y^2) = E[(X-1)^2 u(X-1)] \qquad (\text{Note: } u(x)^2 = u(x))$$

$$= \sum_{k=1}^{\infty} (k-1)^2 p_k$$

$$= \sum_{k=1}^{\infty} (k^2 - 2k + 1)\,p_k$$

$$= E(X^2) - 2E(X) + 1 - p_0$$

6. Let $X$ be the number of children they should have until they have one of each sex. The range of $X$ is $R_X = \{2, 3, 4, \ldots\}$.

Let $B$ be the event that the $1^{st}$ child is a boy. For $i \in R_X$, we have

$$P(X=i) = P(X=i \mid B)\,P(B) + P(X=i \mid B^c)\,P(B^c)$$

$$= \left(\tfrac{1}{2}\right)^{i-2}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^{i-2}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$= \left(\tfrac{1}{2}\right)^{i-1}$$

$$E(X) = \sum_{i=2}^{\infty} i\left(\tfrac{1}{2}\right)^{i-1}$$

$$= \sum_{i=1}^{\infty} i\left(\tfrac{1}{2}\right)^{i-1} - 1$$

$$= \frac{1}{(1-1/2)^2} - 1 \qquad \text{see formula (6.53)}$$

$$= 3$$

7. a) In this case, $X$ is a geometric RV with parameter $p = 1/m$. The PMF of $X$ is

$$p(x) = P(X=x) = \left(\frac{m-1}{m}\right)^{x-1}\frac{1}{m}, \quad x \in R_X = \{1, 2, \ldots\}$$

The results of Th. 6.11 can be applied directly (make sure you understand the proof):

$$E(X) = \frac{1}{p} = m$$

$$Var(X) = \frac{1-p}{p^2} = m(m-1)$$

b) In this case, $R_X = \{1, 2, \ldots, m\}$. Let $A_i$ be the event of selecting the wrong key at the $i^{th}$ trial. For $k \in R_X$, we have

$$p(k) = P(X = k)$$
$$= P(A_1 A_2 \ldots A_{k-1} A_c^k)$$

Using the theorem of multiplication, we have

$$p(k) = P(A_c^k | A_1 \ldots A_{k-1}) P(A_{k-1} | A_1 \ldots A_{k-2}) \ldots P(A_2|A_1) P(A_1)$$

$$= \frac{1}{m - \cancel{k+1}} \cdot \frac{\cancel{m-k+1}}{m - \cancel{k+2}} \cdot \ldots \cdot \frac{\cancel{m-2}}{\cancel{m-1}} \cdot \frac{\cancel{m-1}}{m}$$

$$= \frac{1}{m}$$

$$E(X) = \sum_{k=1}^{m} k\, p(k) = \frac{m+1}{2}$$

$$E(X^2) = \sum_{k=1}^{m} k^2 p(k) = \ldots = \frac{(m+1)(2m+1)}{6}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{m^2-1}{12}$$