ECSE 305, W09
Assignment #2, Solutions

1. (a) $S = \{ppp, ppf, pfp, fpp, pff, fpf, ffp, fff\}$

(b) $Z_F = \{ppf, pff, fpf, fff\}$
$X_p = \{ppp, ppf, pfp, pff\}$

(c) No, $Z_F \cap X_p = \{ppf, pff\} \neq \emptyset$

(d) Assuming that S is an equiprobable space, we have

$$\begin{aligned} P(Z_F \cup X_p) &= P(Z_F) + P(X_p) - P(Z_F \cap X_p) \\ &= 4/8 + 4/8 - 2/8 \\ &= 3/4 \end{aligned}$$

(e) $C = \{ppp, ppf, pfp, fpp\}$
$D = \{pff, fpf, ffp, fff\}$

(f) Yes, $C \cap D = \emptyset$

(g) $$\begin{aligned} P(C \cup D) &= P(C) + P(D), && \text{Axiom 3} \\ &= 4/8 + 4/8 \\ &= 1, && \text{Note that here, } C \cup D = S \end{aligned}$$

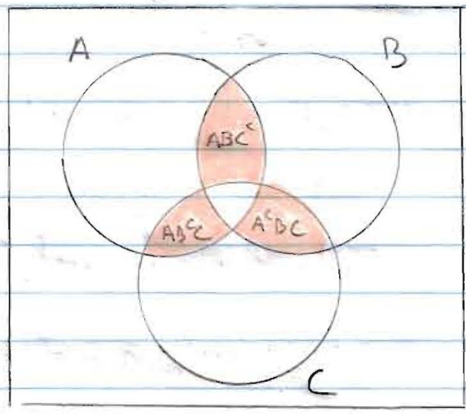
2. (a) Define $D = \{$exactly two of events A, B, C happen$\}$.
Observe that

$$D = ABC^c \cup AB^cC \cup A^cBC$$

where $ABC^c \equiv A \cap B \cap C^c$. Also, the three events ABC^c, AB^cC and A^cBC are mutually exclusive. For example: $(ABC^c) \cap (AB^cC) = ABB^cCC^c = \emptyset$.
Using Axiom and then Theorem 3.5, we have

$$\begin{aligned} P(D) &= P(ABC^c) + P(AB^cC) + P(A^cBC) \\ &= P(AB) - P(ABC) + P(AC) - P(ABC) + P(BC) - P(ABC) \\ &= P(AB) + P(AC) + P(BC) - 3P(ABC) \end{aligned}$$

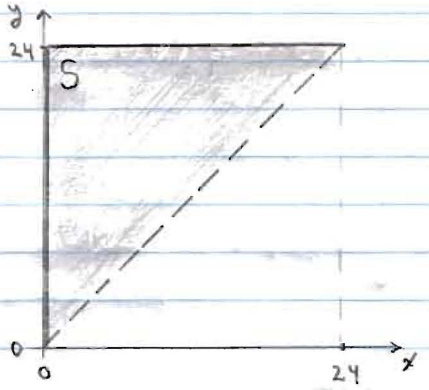
The corresponding situation is illustrated below with a Venn's diagram below, where the shaded area corresponds to D

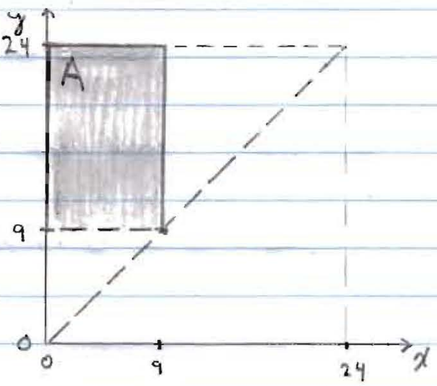


(b) S1 is false; Consider rolling a fair die and let $A = B = C = \{1, 2\}$. Then $P(A) = P(B) = P(C) = 1/2$, $P(A) + P(B) + P(C) = 1$ but A, B, C are not mutually exclusive

S2 is false; Let $A = B = C = S$. Then $A \cup B \cup C = S = 1$; $P(A \cup B \cup C) = P(S) = 1$, but A, B, C are not mutually exclusive.

3. (a) For convenience, let $x = T_1$ and $y = T_2$, both measured in hours. The sample space is $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < y < 24\}$.

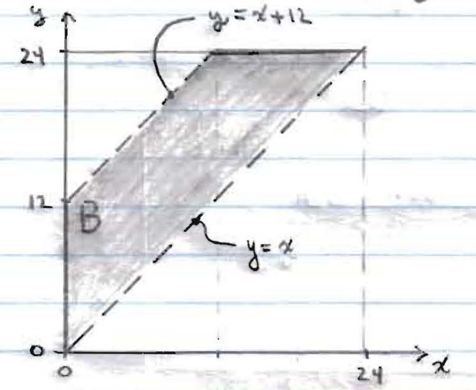


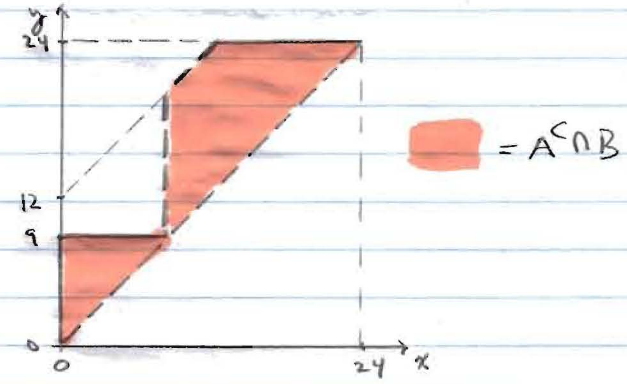


(b) Student is awake at 9 iff $T_1 \leq 9$ and $T_2 > 9$. The corresponding region is shown above. Assuming a uniform probability space over S, we have

$$P(A) = \frac{\text{Area}(A)}{\text{Area}(S)} = \frac{15}{32}$$

(c) Student is asleep more than he is awake iff $T_2 - T_1 < 12$. The region is sketched below.





$$P(B) = \frac{\text{Area }(B)}{\text{Area }(S)} = \frac{3}{4}$$

(d) The region $A^c \cap B$ is shown in orange. This corresponds to: "Student is asleep at 9 o'clock and is asleep more than he is awake".

4. $S = \{x \in \mathbb{R} : -1 \leq x \leq 1\} = [-1, 1]$
$A = \{x \in S : x < 0\} = [-1, 0)$ (Note: we only consider those $x \in S$)
$B = \{x \in S : |x - .5| < 1\} = (-.5, 1]$
$C = (.75, 1]$

(a) $P(B) = \frac{\text{Length}(B)}{\text{Length}(S)} = \frac{3}{4}$

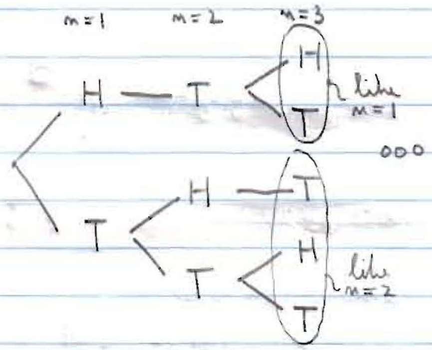
$A \cap B = (-.5, 0) \implies P(A \cap B) = \frac{1}{4}$

$A \cap C = \emptyset \implies P(A \cap C) = 0$

(b) $A \cup B = S \implies P(A \cup B) = 1$

$P(A \cup C) \overset{A3}{=} P(A) + P(C) = 1 + \frac{1}{8} = \frac{5}{8}$

5. Let ϕ_n denote the number of possible runs with no successive heads out of n tosses. To understand the progression of ϕ_n, consider the tree diagram below which illustrate the possible sequences of H and T with no successive H:



We note that $\phi_1 = 2$ and $\phi_2 = 3$. We also observe that $\phi_3 = \phi_1 + \phi_2 = 5$. In fact, this is a general property of this tree which can be shown as follows: Consider the runs with no successive heads of length $n \geq 3$; such runs end on H or T. If they end in T, they can be generated by adding a T to the end of all such runs of length $n-1$. If they end in H, there must be a T before that, so they are generated by adding TH to the end of all such runs of length $n-2$. Therefore

$$\phi_n = \phi_{n-1} + \phi_{n-2} \qquad (1)$$

In fact, we can generate the exact sequence of numbers ϕ_n of interest here, i.e. 2, 3, 5, 8, 11, ... by considering (1) together with initial condition $\phi_{-1} = \phi_0 = 1$. The resulting sequence ϕ_n is the so-called <u>Fibonacci sequence</u>, for which a closed form expression can be found using z-transform techniques. Finally, the desired answer to the problem is

$$P_n = \frac{\phi_n}{2^n} .$$

6\. (a) In total, there are m^n ways of arranging the particles in any of the boxes. There are $n!$ ways of arranging the n particles in the selected boxes. Therefore

$$P_{MB} = \frac{n!}{m^n} \qquad (1)$$

(b) This amounts to the selection of n boxes (one for each particle), with replacement and without ordering, from a set of $m > n$ boxes. In total, the number of such possibilities is (see Chapter 2 in Notes) $(n+m-1)!/n!(m-1)!$, with only one of them corresponding to the desired situation. Therefore

$$P_{BE} = \frac{n!\,(m-1)!}{(n+m-1)!} \qquad (2)$$

<u>Note</u>: Representing the particles by x and the box adjacent walls by |, you can think of an observation as an arrangement of n identical x and $m-1$ identical balls. For example, in the case $n=2$ and $m=4$, the observation

x|x||

corresponds to 1 particle in each of boxes 1 and 2 and no particles in boxes 3 and 4.

(c) In total, there are $\binom{m}{n}$ ways of choosing n boxes among m boxes. Therefore

$$P_{FD} = \frac{1}{\binom{m}{n}} = \frac{n!\,(m-n)!}{m!} \qquad (3)$$

Expressions such as (1)-(3) are used in statististical mechanics to determine the energy distribution of particles as a function of temperature.

7. There are different ways of approaching this problem. Assume that we have 40 distinguishable parts (20 short, and 20 long ones). Let S denote the collection of all possible lists of 20 pairs of 2 parts.[1] Selecting one part at random, there are 39 ways of pairing it. Selecting one of the remaining 38 parts at random, there are 37 ways of pairing it, etc. Therefore

$$N(S) = 39 \cdot 37 \cdot 35 \cdot \ldots \cdot 3 \cdot 1$$

Let A denote the event that long parts are all paired with short ones. Select a long part, there are 20 ways of pairing it. Select one of the remaining long parts, now there are 19 ways of pairing it, etc. Therefore

$$N(A) = 20 \cdot 19 \cdot \ldots \cdot 2 \cdot 1 = 20!$$

$$P(A) = \frac{N(A)}{N(S)} = \frac{20!}{39 \cdot 37 \cdot 35 \cdot \ldots \cdot 3 \cdot 1}$$

Let $B = \{$the new sticks are exactly the same as the old ones$\}$. Here $N(B) = 1$ and

$$P(B) = \frac{1}{N(S)} = \frac{1}{39 \cdot 37 \cdot 35 \cdot \ldots \cdot 3 \cdot 1}$$

8. There are $\binom{52}{14}$ ways of randomly choosing 14 cards. There are $\binom{13}{2}$ ways of choosing 2 hearts, $\binom{13}{3}$ ways of choosing 3 diamonds, etc. Therefore

$$P = \frac{\binom{13}{2}\binom{13}{3}\binom{13}{4}\binom{13}{5}}{\binom{52}{14}}$$

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[1] An outcome, i.e. an element of S, is a set of 20 unordered pairs formed from 40 parts, i.e. $L1, \ldots L20, S1, \ldots S20$. An example of such a list is $\{\underbrace{\{L1, S19\}}_{\text{pair: }1}, \underbrace{\{L2, L14\}}_{2}, \underbrace{\{S7, S20\}}_{3}, \ldots, \underbrace{\{L11, S17\}}_{20}\}$

9. Let e_i denote the envelope in which photo i is placed. Then $S = \{(e_1, \dots, e_{10}) : e_i \in \{1, 2, \dots, 6\}\}$ and $N(S) = 6^{10}$. Let A_i denote the event that envelope i is empty. We note that $N(A_i) = 5^{10}$, $N(A_i \cap A_j) = 4^{10}$ for $i \neq j$, etc.

We want to find the probability

$$P = P(A_1^c \cap A_2^c \cap \dots \cap A_6^c)$$

$$= P((A_1 \cup A_2 \cup \dots \cup A_6)^c)$$

$$= 1 - P(A_1 \cup A_2 \cup \dots \cup A_6)$$

Using the following extension of Theorem 3.4 (also called inclusion - exclusion principle):

mean $A_i \cap A_j$

$$P(A_1 \cup \dots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \dots + (-1)^{n-1} P(A_1 A_2 \dots A_n)$$

we obtain

$$P = 1 - \left\{ \binom{6}{1}\frac{5^{10}}{6^{10}} - \binom{6}{2}\frac{4^{10}}{6^{10}} + \binom{6}{3}\frac{3^{10}}{6^{10}} - \binom{6}{4}\frac{2^{10}}{6^{10}} + \binom{6}{5}\frac{1^{10}}{6^{10}} - \binom{6}{6}\frac{0^{10}}{6^{10}} \right\}$$

$$= 0.2718$$