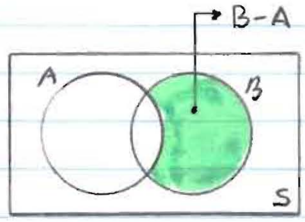
ECSE 305, W09
Assignment #1, Solutions

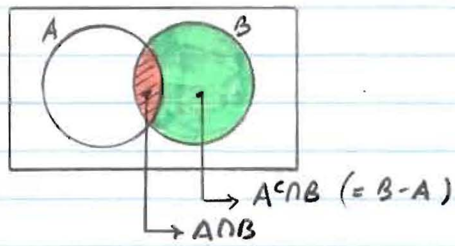
1. (a) C, E
   (b) D, E
   (c) A, B, D
   (d) None

2. (a) Infinite, countable
   (b) Infinite, uncountable
   (c) Infinite, countable
   (d) Finite
   (e) Finite (empty set is finite)

3. (a) $A \cup (B-A) = A \cup (B \cap A^c)$
$= (A \cup B) \cap (A \cup A^c)$
$= (A \cup B) \cap S$
$= A \cup B$



(b) $(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B$
$= S \cap B$
$= B$



Note: you can also prove this type of relations using logical arguments but this is more tedious (see Class Notes, p.22)

4. Observe that $1/(i+1) < 1/i$ and $\lim_{i\to\infty} 1/i = 0$.

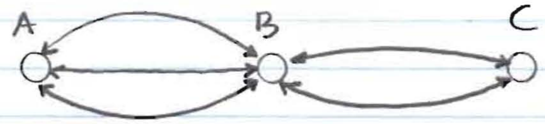
A_i: decreasing sequence, $\lim_{i\to\infty} A_i = \{0\}$

B_i: decreasing sequence, $\lim_{i\to\infty} B_i = \{0\}$

C_i: increasing sequence, $\lim_{i\to\infty} C_i = (0,1)$

D_i: increasing sequence, $\lim_{i\to\infty} D_i = [0,1)$

5.



(a) There are 3 ways to go from A to B, and 2 ways to go from B to C. Hence $n = 3 \cdot 2 = 6$

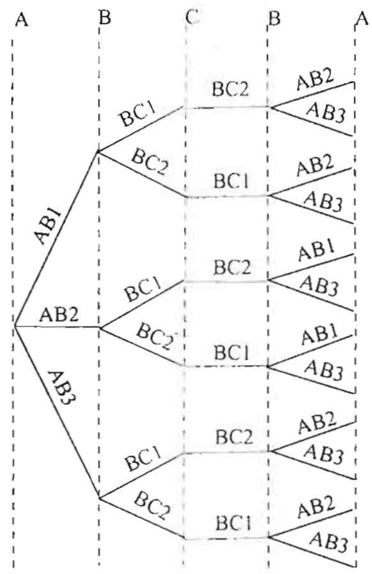
(b) There are 6 ways to go from A to C by way of B and 6 ways to return. Hence $n = 6 \cdot 6 = 36$

(c) The situation can be summarized as follows:

$$A \xrightarrow{3} B \xrightarrow{2} C \xrightarrow{1} B \xrightarrow{2} A$$

Note that since a bus line is not to be used more than once, there is only 1 way to go back from C to B. The final answer is $n = 3 \cdot 2 \cdot 1 \cdot 2 = 12$

(d) Naming the bus lines between A and B as (AB1, AB2, AB3), and the bus lines between B and C as (BC1, BC2), the tree diagram will be:



6. (a) $n_a = 6! = 720$

(b) There are 2 ways to distribute them according to sex: BBBGGG or GGGBBB. In each case, the boys can sit in $3! = 6$ ways and similarly for the girls. Thus, altogether, there are $n_b = 2 \cdot 3! \cdot 3! = 72$ possibilities.

(c) There are 4 ways to distribute them according to sex: GGGBBB, BGGGBB, BBGGGB and BBBGGG. Thus, proceeding as in part (b), there are $n_c = 4 \cdot 3! \cdot 3! = 144$ ways.

(d) For part (a), when the group sits in a circle, all the "circular shifts" of each permutation are the same. For example, numbering the boys and girls, permutations $B_1B_2B_3G_1G_2G_3$, $G_3B_1B_2B_3G_1G_2$, etc. are the same. For each "linear" permutation, there are 6 circular shifts that correspond to the same "circular" permutation. Thus

$$n_a' = \frac{6!}{6} = 120$$

Following the same analysis, the answers must be divided by 2 for part (b) and by 4 for part (c):

$$n_b' = \frac{2 \cdot 3!\, 3!}{2} = 36$$

$$n_c' = \frac{4 \cdot 3!\, 3!}{4} = 36$$

7. (a) This concerns combinations, not permutations, since order does not count in a committee. There are "12 choose 4" such committees. That is

$$n = C(12,4) = \binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495$$

(b) The 2 boys can be chosen from the 9 boys in $\binom{9}{2}$ ways. The 2 girls can be chosen from the 3 girls in $\binom{3}{2}$ ways. Thus

$$n = \binom{9}{2}\binom{3}{2} = \frac{9 \cdot 8}{2 \cdot 1} \cdot \frac{3 \cdot 2}{2 \cdot 1} = 108$$

(c) $\quad n = \binom{9}{3}\binom{3}{1} = 252$

(d) There are 3 cases: 1 girl, 2 girls or 3 girls. Therefore

$$n = \binom{9}{3}\binom{3}{1} + \binom{9}{2}\binom{3}{2} + \binom{9}{1}\binom{3}{3} = 252 + 108 + 9 = 369$$

Alternatively, one can remove the number of committees with no girl, from the total number of committees in (a):

$$n = \binom{12}{4} - \binom{9}{4} = 495 - 126 = 369$$

8. This problem concerns permutations with repetition. There are 9 letters, of which 2 are M, 2 are T and 2 are E. Therefore (Theorem 2.7)

$$n = \frac{9!}{2!\,2!\,2!} = 45360$$

When the letters C and E are chosen as first an last, respectively, we are left with 7 letters, of which 2 are M and 2 are T. Thus

$$n = \frac{7!}{2!\,2!} = 1260$$

9. There are 2 ways of distributing the 10 persons according to country: CFCFCFCFCF or FCFCFCFCFC. In each case, the 5 persons from Canada can be arranged in 5! ways and similarly for the 5 persons from France. Therefore $n = 2 \cdot 5! \cdot 5! = 28800$ (see problem 6)

10. We use mathematical induction. The theorem is true for $n=0$. That is: $(x+y)^0 = 1 = \sum_{i=0}^{0} \binom{0}{i} x^{0-i} y^i$. Now, assuming that the theorem holds for any given integer $n \geq 1$, we must show that it also holds for $n+1$:

$$
\begin{aligned}
(x+y)^{n+1} &= x(x+y)^n + y(x+y)^n \\
&= x \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i + y \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i \\
&= x^{n+1} + \sum_{i=1}^{n} \binom{n}{i} x^{n+1-i} y^i + \sum_{i=0}^{n-1} \binom{n}{i} x^{n-i} y^{i+1} + y^{n+1} \\
&= x^{n+1} + \sum_{i=1}^{n} \left[\binom{n}{i} + \binom{n}{i-1} \right] x^{n+1-i} y^i + y^{n+1} \\
&= x^{n+1} + \sum_{i=1}^{n} \binom{n+1}{i} x^{n+1-i} y^i + y^{n+1} \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} x^{n+1-i} y^i
\end{aligned}
$$

where we have used the identity $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$ (see Theorem 2.9).

The expression $(x+y)^n$ is a polynomial with $n+1$ terms of the type $x^{n-i} y^i$ for $i = 0, 1, \ldots, n$. For a given monomial $x^{n-i} y^i$, there are $\binom{n}{i}$ ways of chosing the i "y" from the products $(x+y) \ldots (x+y)$. For example:

$$
\begin{aligned}
(x+y)^2 &= (x+y)(x+y) \\
&= x^2 + (xy + yx) + y^2 \\
&= x^2 + \binom{2}{1} xy + y^2
\end{aligned}
$$