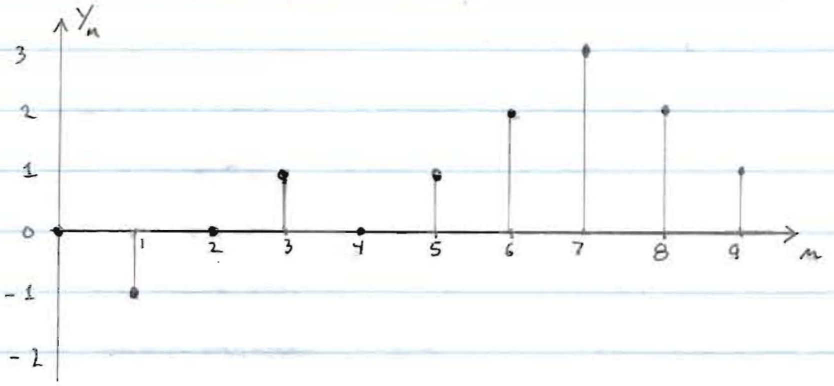
ECSE 305, W09

Assignment #10, Solutions

1. a) • $Y_n$ is discrete-time: time argument $n$ takes value in countably infinite set $T = \{0, 1, 2, \ldots\}$.

• $Y_n$ is discrete-space: the possible values of $Y_n$ are given by the countably infinite set $\Omega = \{0, \pm 1, \pm 2, \ldots\}$.

b)




c) We first note that for any $i \geq 1$:

$$E(X_i) = (1)p + (-1)q = p - q$$
$$E(X_i^2) = (1)^2 p + (-1)^2 q = p + q = 1$$
$$Var(X_i) = E(X_i^2) - E(X_i)^2$$
$$= 1 - (p-q)^2 = \ldots = 4pq$$

Thus, for $n \geq 1$, we have

$$\mu_Y(n) \triangleq E(Y_n) = E\left(\sum_{i=1}^{n} X_i\right)$$
$$= \sum_{i=1}^{n} E(X_i) = n(p-q)$$

note: the $X_i$ are independent

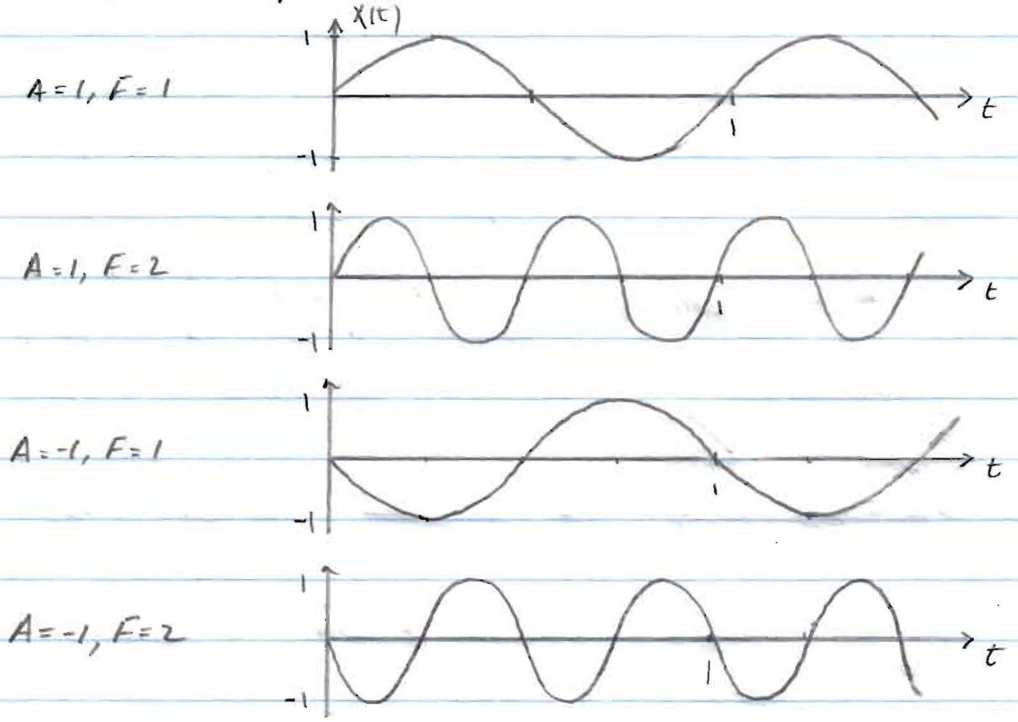
$$\sigma_Y^2(n) \triangleq Var(Y_n) = Var\left(\sum_{i=1}^{n} X_i\right)$$
$$= \sum_{i=1}^{n} Var(X_i) = 4npq$$

For $n = 0$, $Y_0 = 0$ and so $E(Y_0) = 0$, $Var(Y_0) = 0$. The above formulas also apply to $n = 0$.

2. a) • Continuous-time: $t \in T = [0, \infty)$

• Continuous-space: for an arbitrary $t$, $X(t) \in \Omega = [-1, 1]$

b) There are 4 possibilities:




c) We first note that $E(A) = (1)\frac{1}{2} + (-1)\frac{1}{2} = 0$. Since A and F are independent, we have

$$\begin{aligned}
\mu_X(t) &\triangleq E(X(t)) \\
&= E(A \sin(2\pi F t)) \\
&= \cancel{E(A)}\, E(\sin 2\pi F t) \\
&= 0, \quad \text{all } t \in T
\end{aligned}$$

3. a) From Theorem 13.2, we have

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, R_X(\tau - u_1 + u_2)\, du_1\, du_2$$

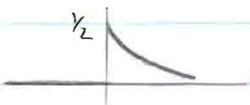
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1) h(u_2)\, \delta(u_2 - (u_1 - \tau))\, du_1 du_2$$

$$= \int_{-\infty}^{\infty} h(u_1)\, h(u_1 - \tau)\, du_1 \qquad (= h(\tau) * h(-\tau))$$

$$\tau \geq 0: \quad R_Y(\tau) = \int_{\tau}^{\infty} e^{-u_1}\, e^{-(u_1-\tau)}\, du_1$$

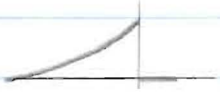
$$= e^{\tau} \int_{\tau}^{\infty} e^{-2u_1}\, du_1$$

$$= e^{\tau} \left( \frac{e^{-2u_1}}{-2} \right) \Big|_{\tau}^{\infty} = \frac{1}{2} e^{-\tau}$$



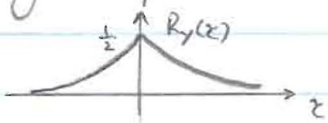

$$\tau \leq 0: \quad R_Y(\tau) = \int_{0}^{\infty} e^{-u_1}\, e^{-(u_1-\tau)}\, du_1$$

$$= \quad \cdots \quad = \frac{1}{2} e^{\tau}$$



These two cases can be combined into a single expression:

$$\boxed{R_Y(\tau) = \frac{1}{2} e^{-|\tau|}, \qquad \tau \in \mathbb{R}}$$




b) The frequency response of the filter is

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(1+j\omega)t}\, dt = \frac{1}{1+j\omega}$$

The PSD of the input process is

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau}\, d\tau$$

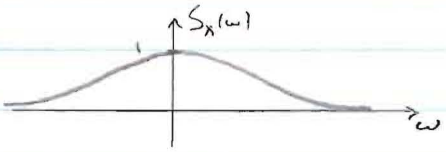
$$= \int_{-\infty}^{\infty} \delta(\tau)\, e^{-j\omega\tau}\, d\tau = 1$$

The PSD of the output process is (Theorem 13.3)

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$$

$$= \left|\frac{1}{1+j\omega}\right|^2$$

$$= \frac{1}{1+\omega^2}$$




4. $$\mu_X(t) \triangleq E[X(t)]$$
$$= E[A \cos(\omega t + B)]$$
$$= \cancel{E[A]}\, E[\cos(\omega t + B)] \quad \text{(because A and B are independent)}$$
$$= 0 \quad \text{(because } A \sim N(0,1))$$

$$R_X(t,u) \triangleq E[X(t)X(u)]$$
$$= E[A^2 \cos(\omega t + B) \cos(\omega u + B)]$$
$$= E[A^2]\, E[\cos(\omega t + B)\cos(\omega u + B)] \qquad (E(A^2) = 1)$$
$$= \frac{1}{2} E[\cos(\omega(t-u)) + \cos(\omega(t+u) + 2B)]$$
$$= \frac{1}{2}\cos(\omega(t-u)) + \frac{1}{2} E[\cos(\omega(t+u) + 2B)]$$
$$= \frac{1}{2}\cos(\omega(t-u)) + \frac{1}{2}\underbrace{\cancel{\int_0^{2\pi} \cos(\omega(t+u) + 2b)}}_{0}\, \underbrace{\frac{1}{2\pi}}_{\text{PDF of B}}\, db$$
$$= \frac{1}{2}\cos(\omega(t-u))$$

Since $\mu_X(t)$ is a constant and $R_X(t,u)$ is only a function of the difference (or lag) $t-u$, we conclude that $X(t)$ is WSS.

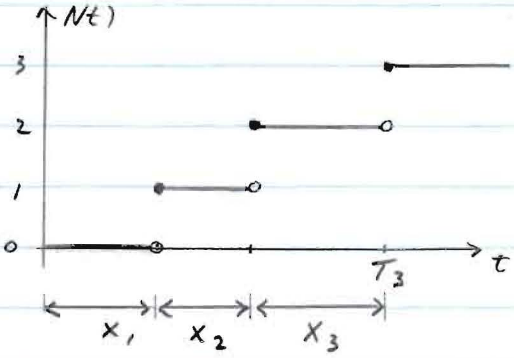
5. $R_Y(t_1,t_2) \triangleq E[Y(t_1)Y(t_2)]$

$$= E\left[\int_{-\infty}^{\infty} h(u_1)X(t_1-u_1)\,du_1 \int_{-\infty}^{\infty} h(u_2)X(t_2-u_2)\,du_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)\,E[X(t_1-u_1)X(t_2-u_2)]\,du_1\,du_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)\,R_X(\underbrace{(t_1-u_1)-(t_2-u_2)}_{=\,t_1-t_2-u_1+u_2})\,du_1\,du_2$$

6. a) Let $X_i$ denote the interarrival time of the $i^{th}$ patient. RVs $X_i$ are exponentially distributed with parameter $\lambda = .1\,\text{min}^{-1}$. We seek




$$E(T_3) = E(X_1+X_2+X_3)$$
$$= E(X_1)+E(X_2)+E(X_3)$$
$$= 3\,\tfrac{1}{\lambda}$$
$$= 30 \text{ min}$$

b) Observe that $T_3 > 60$ min iff $N(60) \leq 2$. Also $N(60)$ is a Poisson RV with parameter $60\lambda = 6$. Hence

$$P(T_3 > 60) = P(N(60) \leq 2)$$

$$= \sum_{i=0}^{2} P(N(60) = i)$$

$$= \sum_{i=0}^{2} \frac{6^i}{i!} e^{-6}$$

$$= (1+6+18)e^{-6} \simeq 0.062$$

7. a) It can be seen that

$$P(X=1, Y=-1) = 0 \neq P(X=1)P(Y=-1) = \frac{1}{4} \cdot \frac{17}{48}.$$

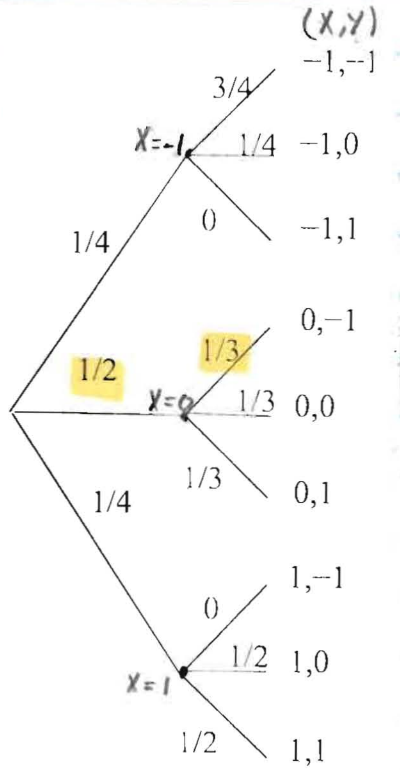
Therefore, X and Y are NOT independent.

b) From the given data, we must first compute the marginal PMF of X, and the conditional PMF of Y given $X=x$, e.g.

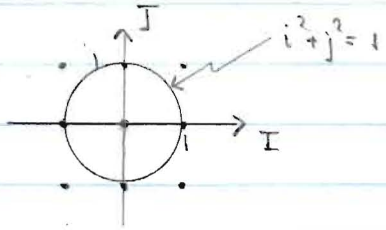
$$p_X(0) = 1/6 + 1/6 + 1/6 = 1/2$$

$$p_{Y|X}(-1|0) = \frac{p(0,-1)}{p_X(0)} = \frac{1/6}{1/2} = 1/3$$

The resulting tree diagram is shown on the right:



8. a) $\displaystyle\sum_{\text{all}(i,j)} p(i,j) = \sum_{i=-1}^{1}\sum_{j=-1}^{1} c\,(i^2+j^2)$




$= c\,(0+4+4\times 2)$

$= 12c \implies c = \frac{1}{12}$

b) $\displaystyle p_J(j) = \sum_{i=-1}^{1} p(i,j)$

$= \frac{1}{12}\left[((-1)^2+j^2) + (0^2+j^2) + (1^2+j^2)\right]$

$= \frac{1}{12}(2+3j^2), \quad j \in \{-1,0,1\}$

$p_{I|J}(i,j) = \dfrac{p(i,j)}{p_J(j)}$

$= \dfrac{\frac{1}{12}(i^2+j^2)}{\frac{1}{12}(2+3j^2)}$

$= \dfrac{i^2+j^2}{2+3j^2}, \quad i,j \in \{-1,0,1\}$

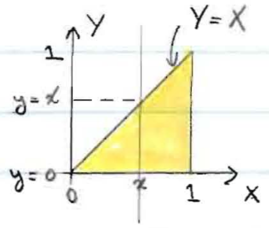
c) $P(|I|=1 \,|\, J=0) = P(I=-1 \,|\, J=0) + P(I=1 \,|\, J=0)$

$= p_{I|J}(-1|0) + p_{I|J}(1|0)$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$

9. $f(x,y) = \begin{cases} 6y, & 0 \leq y \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$




a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$x \in [0,1] \Rightarrow f_X(x) = \int_0^x 6y\, dy = 3x^2$

$x \notin [0,1] \Rightarrow f_X(x) = 0$

b) $f_{Y|X}(y|x)$ is defined whenever $f_X(x) > 0$, i.e. for $x \in (0,1]$. In this case we have

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} 2y/x^2, & 0 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

c) From part b), we have $f_{Y|X}(y|1/2) = 8y$ for $0 \leq y \leq 1/2$.

$$P(Y \leq \tfrac{1}{4} \mid X = \tfrac{1}{2}) = \int_{-\infty}^{1/4} f_{Y|X}(y|1/2)\, dy$$

$$= \int_0^{1/4} 8y\, dy$$

$$= \left(4y^2\right)\Big|_0^{1/4} = \frac{1}{4}$$

d) $E[Y|X=x]$ is defined for $x \in (0,1]$:

$$E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy$$

$$= \int_0^x y\, \frac{2y}{x^2}\, dy$$

$$= \frac{2}{x^2}\left(\frac{y^3}{3}\right)\Big|_0^x = \frac{2}{3}x$$