ECSE-305, Winter 2009
Probability and Random Signals I
Assignment #10

Posted: Thursday, April 2, 2009.

Due: Tuesday, April 14, 2009, 11h00am, MC756.

Notes: Assignments without this cover page will be discarded.

Student #1:

Name:____________________________

ID:______________________________

Student #2:

Name:____________________________

ID:______________________________

Question	Marks
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1. Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with $P(X_n = 1) = p$ and $P(X_n = -1) = q = 1 - p$, for all n. Define

$$Y_n = \sum_{i=1}^{n} X_i \quad n = 1, 2, \ldots$$

and $Y_0 = 0$. The collection of RVs $\{Y_n : n \geq 0\}$ is a random process, called a *random walk*.

   (a) What type of process is Y_n? Identify the index parameter T and the state space Ω.

   (b) Construct a typical realization of Y_n in the case $p = 1/2$. (Hint: use a coin...)

   (c) Find the mean and variance function, i.e. $\mu_Y(n)$ and $\sigma_Y^2(n)$, of the process Y_n.

2. Consider a random process $X(t)$ defined by

$$X(t) = A \sin(2\pi F t), \quad t \geq 0$$

where the amplitude A is a discrete RV with $P(A = 1) = P(A = -1) = 1/2$, and the frequency F is a discrete RV with $P(F = 1) = P(F = 2) = 1/2$.

   (a) What type of process is $X(t)$? Identify the index parameter T and the state space Ω.

   (b) Illustrate all the possible realizations of $X(t)$.

   (c) Find the mean function $\mu_X(t)$ of the process $X(t)$. Assume that RVs A and F are independent.

3. Let $\mathcal{H}$ denote a low pass filter with impulse response

$$h(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Assume that the input to the filter, say $X(t)$, is a WSS process with mean $\mu_X = 0$ and autocorrelation function $R_X(\tau) = \delta(\tau)$.

   (a) Find the autocorrelation function of the output process $Y(t)$.

(b) Find the power spectral density of $Y(t)$.

4. Random process $X(t)$ is defined by

$$X(t) = A\cos(\omega t + B), \quad t \in \mathbb{R}$$

where A is a normal RV with zero-mean and unit variance, B is uniform over $[0, 2\pi)$, A and B are independent, and ω is a deterministic angular frequency. Determine whether or not $X(t)$ is wide sense stationary?

5. Prove theorem 13.2 in the class notes.

6. A dentist's office opens at 9h00am, after which patients arrive according to a Poisson process with rate $\lambda = 0.1$ per minute. The dentist will only start seeing patients when at least 3 of them are in the waiting room.

   (a) Find the expected time at which the dentist will see the first patient.

   (b) What is the probability that the dentist will see the first patient only after 10h00am?

7. Random variables X and Y have a joint PMF described by the following table:

$p(x,y)$	$y=-1$	$y=0$	$y=1$
$x=-1$	3/16	1/16	0
$x=0$	1/6	1/6	1/6
$x=1$	0	1/8	1/8

   (a) Are X and Y independent? Explain.

   (b) Suppose that the values of X and Y are derived from a sequential experiment: first, X is found, then Y is found. Illustrate this experiment by means of a tree diagram and label each branch with the corresponding (numerical) value of the transition probability.

8. Let the joint PMF of discrete RVs I and J be given by

$$p(i,j) = \begin{cases} c(i^2 + j^2), & i = 0, \pm 1 \text{ and } j = 0, \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the constant c.

(b) Find the conditional PMF $p_{I|J}(i|j)$ for $j \in \{0, \pm 1\}$.

(c) Find $P(|I| = 1|J = 0)$.

9. Let X and Y be random variables with joint PDF

$$f(x, y) = \begin{cases} 6y, & 0 \leq y \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal PDF of $f_X(x)$.

(b) Find the conditional PDF $f_{Y|X}(y|x)$. For what values of x is $f_{Y|X}(y|x)$ defined.

(c) Find $P(Y \leq \frac{1}{4}|X = \frac{1}{2})$.

(d) Find the conditional expected value $E[Y|X = x]$.