ECSE 304-305B Solutions to Assignment 9 Winter 2007

Solution 9.1

(a)

If $\rho = 0$, then

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{-\frac{1}{2}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2+\left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-m_1)^2}{2\sigma_1}\right\}\cdot\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{(y-m_2)^2}{2\sigma_2}\right\} \\ &= f_X(x)\cdot f_Y(y). \end{aligned}$$

Hence, X and Y are independent.

(b)

$$\begin{aligned} P[XY>0] &= P[X>0,Y>0]\cup P[X<0,Y<0] \\ &= P[X>0,Y>0]+P[X<0,Y<0]+\underbrace{P[X>0,Y>0]\cap P[X<0,Y<0]}_{=\emptyset}. \end{aligned}$$

Since $m_1 = m_2 = 0$, $\rho = 0$, X and Y are zero mean, independent Gaussian random variables. Hence

$$\begin{aligned} P[X>0,Y>0] &= P[X>0]P[Y>0] = 1/4 \\ P[X<0,Y<0] &= P[X<0]P[Y<0] = 1/4, \end{aligned}$$

and $P[XY > 0] = 1/4 + 1/4 = 1/2$.

(c)

$$\begin{aligned} R_{1,1} &:= E(X-m_1)^2 = \sigma_1^2 \\ R_{2,2} &:= E(Y-m_2)^2 = \sigma_2^2 \\ R_{1,2} &:= E(X-m_1)(Y-m_2) = EXY - m_1m_2 \end{aligned}$$

since $EXY = \rho\sigma_1\sigma_2 + m_1m_2$,

$$R_{1,2} = \rho\sigma_1\sigma_2 = R_{2,1}.$$

then for $n = 2$,

$$\begin{aligned}(2\pi)^{n/2}|\det R|^{1/2} &= 2\pi \left|\det \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right|^{1/2} \\ &= 2\pi \left(\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2\right)^{1/2} \\ &= 2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}.\end{aligned}$$

Solution 9.2

(i)

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi|\Sigma_{X,Y}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x,y)\Sigma_{X,Y}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right\} \\ &= \frac{1}{2\pi\sqrt{3}} \exp\left\{-\frac{1}{3}(x^2+y^2-xy)\right\}\end{aligned}$$

(ii) From $\Sigma_{X,Y}$, we know $\rho_{X,Y} = \frac{1}{2}$.

(iii)

$$\begin{aligned}\mu_{U,V} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mu_{X,Y} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \\ \Sigma_{U,V} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\Sigma_{X,Y}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

(iv) From $\Sigma_{U,V}$, we know $\rho_{U,V} = 0$.

(v) Since

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N\left(\mu_{U,V}, \Sigma_{U,V}\right),$$

we have

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi|\Sigma_{U,V}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\left[(u,v)-\mu_{U,V}^T\right]\Sigma_{U,V}^{-1}\left[\begin{pmatrix} u \\ v \end{pmatrix} - \mu_{U,V}\right]\right\} \\ &= \frac{1}{2\pi\sqrt{3}} \exp\left\{-\frac{(u-2)^2}{6} - \frac{(v+1)^2}{2}\right\}\end{aligned}$$

Solution 9.3

From

$$\begin{aligned}\frac{1}{2\pi}\exp\left\{-\frac{z^2-2zw+2w^2}{2}\right\} &= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(z,w)\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}z\\ w\end{pmatrix}\right]\\ &= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(z,w)\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}z\\ w\end{pmatrix}\right],\end{aligned}$$

we know

$$\mu_{Z,W} = \begin{bmatrix}0\\ 0\end{bmatrix}, \qquad \text{and} \qquad \Sigma_{Z,W} = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}.$$

Since

$$\begin{bmatrix}X\\ Y\end{bmatrix} = \begin{bmatrix}2 & -3\\ 0 & 4\end{bmatrix}\begin{bmatrix}Z\\ W\end{bmatrix},$$

$$\begin{aligned}\mu_{X,Y} &= \begin{bmatrix}2 & -3\\ 0 & 4\end{bmatrix}\mu_{Z,W} = \begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}m_1\\ m_2\end{bmatrix}, \qquad \text{and}\\ \Sigma_{X,Y} &= \begin{bmatrix}2 & -3\\ 0 & 4\end{bmatrix}\Sigma_{Z,W}\begin{bmatrix}2 & -3\\ 0 & 4\end{bmatrix}^T = \begin{bmatrix}5 & -4\\ -4 & 16\end{bmatrix} = \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix} = R,\end{aligned}$$

with $\sigma_1 = \sqrt{5}$, $\sigma_2 = 4$, $\rho = -\frac{1}{\sqrt{5}}$, and $\det R = 64$.

Hence

$$\begin{aligned}&f_{X,Y}(x,y)\\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]}{2(1-\rho^2)}\right\}\\ &= \frac{1}{16\pi}\exp\left\{-\frac{16x^2+8xy+5y^2}{128}\right\}\end{aligned}$$