ECSE 304-305B   Assignment 7   Winter 2007

Return by 12.00 pm, 12th March

7.1

Consider a sequence of independent random variables $\{X_k; 1 \le k\}$ such that X_k is uniformly distributed on the interval $[-k, k]$. Define:

$$Y_n = \sum_{k=1}^{n} X_k, \qquad Z_n = \sum_{k=1}^{n} (-1)^k X_k$$

Show by the use of characteristic functions that Y_n and Z_n have identical distributions for all $\{n; 1 \le n\}$.

Solution:

Since X_i are independent random variables and uniformly distributed on the interval $[-k, k]$, the characteristic functions are given by

$$\Phi_{X_i}(w) = \int_{-k}^{k} e^{jwx} \frac{1}{2k} dx = \frac{e^{jwk} - e^{-jwk}}{2kjw},$$

which are even functions since $\Phi_{X_i}(w) = \Phi_{X_i}(-w)$.

Given $Y_n = \sum_{k=1}^{n} X_k$,

$$\begin{aligned} \Phi_{Y_n}(w) &= E(\exp^{jwY_n}) = E(\exp^{jw(X_1+X_2+\cdots+X_n)}) \\ &= E(\exp^{jwX_1})E(\exp^{jwX_2})\cdots E(\exp^{jwX_n}) \\ &= \Phi_{X_1}(w)\Phi_{X_2}(w)\cdots\Phi_{X_n}(w). \end{aligned}$$

Given $Z_n = \sum_{k=1}^{n} (-1)^k X_k$,

$$\begin{aligned} \Phi_{Z_n}(w) &= E(\exp^{jwZ_n}) = E(\exp^{jw(-X_1+X_2+\cdots+(-1)^n X_n)}) \\ &= E(\exp^{-jwX_1})E(\exp^{jwX_2})\cdots E(\exp^{(-1)^n jwX_n}) \\ &= \Phi_{X_1}(-w)\Phi_{X_2}(w)\cdots\Phi_{X_n}((-1)^n w) \end{aligned}$$

Since each Φ_i is an even function of w, we have

$$\Phi_{Z_n}(w) = \Phi_{X_1}(w)\Phi_{X_2}(w)\cdots\Phi_{X_n}(w),$$

that is to say $\Phi_{Z_n}(w) = \Phi_{Y_n}(w)$.

And because distribution functions are in one to one relation with characteristic functions, we conclude that Y_n and Z_n have identical distributions.

7.2

(i) Let the exponentially distributed random variable X with parameter $\lambda > 0$ model the waiting time until the random instant at which an event occurs:

$$P(X \leq t) \quad = \quad 1 - e^{-\lambda t} \qquad\qquad t \in R_+$$

Show that X possesses the memoryless property:

$$P(\ X\ >\ t + h \mid X\ >\ t\ )\ =\ P(\ X\ >\ h\ ).$$

This may be interpreted as the waiting process restarting from zero at any given time.

[Hence, if the occurrence of an event is exponentially distributed, the fact that one has waited t seconds for it to happen has no influence on the probability whether you will see the event occur in the next h seconds. (This is viewed as bad by someone in an exponential bus queue; one's investment in waiting is of no value.)]

(ii) Give the characteristic function of the exponential waiting time distribution on $[0, \infty)$ with parameter $\lambda > 0$.

A traveler at Trudeau International Airport must wait in two queues in ler at Trudeau International Airport must wait in two queues in series: first, the traveler must wait at the Check-in queue for his orler must wait at the Check-in queue for his or her airline; this has an exponentially distributed waiting time T_λ, with parameter $\lambda > 0$; second, the traveler must wait in a queue in the ler must wait in a queue in the Security Zone with an exponentially distributed waiting time T_μ, with parameter $\mu > 0$.
It is assumed that T_λ and T_μ are independent random variables.

(iii) What is the characteristic function of the total waiting time $T_\lambda + T_\mu$?

(iv) Find the characteristic function of $2T_\lambda$.

(v) By use of characteristic functions, or otherwise, show whether the density of $T_\lambda + T_\mu$ with $\mu = \lambda$ is the same as that of $2T_\lambda$.

(vi) Find the second moment of $T_\lambda + T_\mu$.

Solution:

(i)

$$\begin{aligned} P(X > t+h | X > t) &= \frac{P(X > t+h, X > t)}{P(X > t)} \\ &= \frac{P(X > t+h)}{P(X > t)} \\ &= \frac{e^{-\lambda(t+h)}}{e^{-\lambda(t)}} \\ &= e^{-\lambda h} \\ &= P(X > h). \end{aligned}$$

(ii)

$$\Phi_{T_\lambda}(w) = \int_{-\infty}^{\infty} e^{jwt} \cdot f_t(t)\, dt = \int_0^{\infty} e^{jwt} \cdot \lambda \cdot e^{-\lambda t} dt = \frac{\lambda}{\lambda - jw}.$$

(iii) Similar to part (i), we have

$$\Phi_{T_\mu}(w) = \frac{\mu}{\mu - jw}.$$

Since T_λ and T_μ are independent random variables, the characteristic function of the total waiting time $T_\lambda + T_\mu$ is

$$\Phi_{T_\lambda + T_\mu}(w) = \Phi_{T_\lambda}(w) \cdot \Phi_{T_\mu}(w) = \frac{\lambda\mu}{(\lambda - jw)(\mu - jw)}.$$

(iv)

$$\Phi_{2T_\lambda}(w) := E\left[e^{jw2T_\lambda}\right] = E\left[e^{j(2w)T_\lambda}\right] = \Phi_{T_\lambda}(2w) = \frac{\lambda}{\lambda - j(2w)}.$$

(v)

$$\begin{aligned} \Phi_{T_\lambda+T_\mu}(w)\Big|_{\mu=\lambda} &= \frac{\lambda\mu}{(\lambda-jw)(\mu-jw)}\Big|_{\mu=\lambda} = \frac{\lambda^2}{(\lambda-jw)^2} \\ &= \frac{\lambda^2}{\lambda^2-2j\lambda\mu-w^2} \neq \Phi_{2T_\lambda}(w). \end{aligned}$$

Hence their density functions could not be the same, since

$$f_x(x) = (1/2\pi)\int_{-\infty}^{\infty} \Phi_x(w)\cdot\exp(-jwx)\,dw.$$

(vi)

$$\begin{aligned} EX^2 &= \frac{1}{j^2}\frac{d^2}{d\,w^2}\Phi_x(w)\Big|_{w=0} \\ &= \frac{2}{\mu^2}+\frac{2}{\lambda^2}+\frac{2}{\mu\lambda}. \end{aligned}$$

7.3

For a random variable X with a probability density function $f_X(\cdot)$, let $Y = g(X)$, and consider the four cases:

(a) $g(x) = -x$, where X is uniformly distributed on $[-1, 1]$,

(b) $g(x) = x^3$, where X is uniformly distributed on $[1, 4]$,

(c) $g(x) = 2|x|$, where X has the Gaussian density $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}exp^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$,

(d) $g(x) = -|x+2|$, where X has the Gaussian density $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}exp^{-\frac{1}{2}\frac{(x-1)^2}{\sigma^2}}$.

Find the formula for the probability density $f_Y(\cdot)$ of Y in each case at values of y for which $\frac{dy}{dx}$ exists and $\frac{dy}{dx} \neq 0$.

Solution:

(a) Given:

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$y = -x$, $x \in [-1, 1]$, implies for $-1 \leq y \leq 1$, $x(y) = -y$.

The derivative for the particular solution is

$$\frac{dx(y)}{dy} = -1.$$

Hence,

$$f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right| = \begin{cases} \frac{1}{2}, & -1 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Given:

$$f_X(x) = \begin{cases} \frac{1}{3}, & 1 \leq x \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leq y \leq 64$,

$$x(y) = \sqrt[3]{y}.$$

The derivative for the particular solution is

$$\frac{dx(y)}{dy} = \frac{1}{3} y^{-\frac{2}{3}}.$$

Hence,

$$f_Y(y) = f_X(x(y)) \left| \frac{dx(y)}{dy} \right| = \begin{cases} \frac{1}{9} y^{-\frac{2}{3}}, & 1 \leq y \leq 64 \\ 0, & \text{otherwise.} \end{cases}$$

(c) Given:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{x^2}{2\sigma^2}}, \forall x.$$

For $y \geq 0$,

$$x_1(y) = \frac{y}{2}, \qquad \text{or} \qquad x_2(y) = -\frac{y}{2}.$$

However $\frac{dy}{dx}$ does not exist at $x = 0$, so $y > 0$.

Hence,

$$\begin{aligned} f_Y(y) &= f_X(x_1(y))\left|\frac{dx_1(y)}{dy}\right| + f_X(x_2(y))\left|\frac{dx_2(y)}{dy}\right| \\ &= \begin{cases} \frac{1}{\sqrt{2\pi}\sigma}\exp^{-\frac{y^2}{8\sigma^2}}, & y > 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(d) Given:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp^{-\frac{(x-1)^2}{2\sigma^2}}, \forall x.$$

For $y \leq 0$, $y = -|y|$.

So

$$x_1(y) = -y - 2, \qquad \text{or} \qquad x_2(y) = y - 2.$$

However $\frac{dy}{dx}$ does not exist at $x = -2$, so $y < 0$.

Hence,

$$\begin{aligned} f_Y(y) &= f_X(x_1(y))\left|\frac{dx_1(y)}{dy}\right| + f_X(x_2(y))\left|\frac{dx_2(y)}{dy}\right| \\ &= \begin{cases} \frac{1}{\sqrt{2\pi}\sigma}(\exp^{-\frac{(y+3)^2}{2\sigma^2}} + \exp^{-\frac{(y-3)^2}{2\sigma^2}}), & y < 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

7.4

Assume each of the independent identically distributed scalar random variables $X_i, 1 \leq i < \infty$, has mean 0 and variance $\sigma^2 = 4$. For $\alpha > 0$, consider the probability of the event :

$$A = \{-\alpha \leq \tfrac{1}{n}\textstyle\sum_{i=1}^{n} X_i \leq \alpha\}$$

(i) Use the Central Limit Theorem, together with the notation $\Phi(x), x \in R$, for the distribution function of a normally distributed $N(0,1)$ random variable, to give a formula for an approximation to the probability that the average $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ lies in the interval $[-\alpha, \alpha]$.

(ii) Let $\alpha = 1$. Use the CLT based formula to find the smallest value of n for which the probability of A is at least: (a) 0.95 and (b) 0.9786. (You may use the fact that for the Gaussian distribution $\Phi(-x) = 1 - \Phi(x), x \in R$, and may use any standard Gaussian distribution table; for instance in the course text this is given on page SG 632.)

Solution:

(i) By CLT,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i}{\sqrt{4}} \approx N(0,1)$$

Therefore,

$$\begin{aligned} P(-\alpha \leq \frac{1}{n} \sum_{i=1}^{n} X_i \leq \alpha) &= P(-\frac{\alpha\sqrt{n}}{2} \leq \frac{1}{\sqrt{4n}} \sum_{i=1}^{n} X_i \leq \frac{\alpha\sqrt{n}}{2}) \\ &\approx \Phi(\frac{\alpha\sqrt{n}}{2}) - \Phi(-\frac{\alpha\sqrt{n}}{2}) \\ &= 2\Phi(\frac{\alpha\sqrt{n}}{2}) - 1 \end{aligned}$$

(ii) When α=1:

(a) $2\Phi(\frac{\sqrt{n}}{2}) - 1 \geq 0.95 \Rightarrow \Phi(\frac{\sqrt{n}}{2}) \geq 0.975 \Rightarrow \frac{\sqrt{n}}{2} \geq 1.96 \Rightarrow n \geq 15.37$

Since n is an integer, it has to be equal to or bigger than 16.

(b) $2\Phi(\frac{\sqrt{n}}{2}) - 1 \geq 0.9786 \Rightarrow \Phi(\frac{\sqrt{n}}{2}) \geq 0.9893 \Rightarrow \frac{\sqrt{n}}{2} \geq 2.3 \Rightarrow n \geq 21.16$

Since n is an integer, it has to be equal to or bigger than 22.

7.5

The random variable X has the Binomial distribution $B(N, \frac{1}{2})$, i.e. it is the sum of N independent Bernoulli $\{+1, -1\}$ valued random variables $\{Y_k; 1 \leq k \leq N\}$ each of which satisfies $P(Y_k = +1) = P(Y_k = -1) = \frac{1}{2}$.

(a) find EX,

(b) show whether $E(X^2)$ has the value N or $\frac{1}{2}N$,

(c) check you answer in (b) in the case $N = 2$,

(d) use the Chebychev inequality to estimate $P(|X - EX| > N)$.

Hint: The Moment Theorem and the characteristic function $Ee^{iX\omega}$ may be used in parts (a) and (b) if you wish.

Solution:

The characteristic function of $Y_k, k = 1, ..., N$ is:

$$\Phi_{Y_k}(\omega) = E(e^{j\omega Y_k}) = 0.5 \cdot e^{j\omega} + 0.5 \cdot e^{-j\omega} = \cos\omega$$

Hence $X = \sum_{k=1}^{N} Y_k$, the characteristic function is:

$$\begin{aligned}\Phi_X(\omega) &= E(e^{j\omega \sum_{k=1}^{N} Y_k}) = E(\prod_{k=1}^{N} e^{j\omega Y_k}) \\ &= \prod_{k=1}^{N} E(e^{j\omega Y_k}) \\ &= \prod_{k=1}^{N} \cos\omega = \cos^N \omega\end{aligned}$$

(a)

$$E(X) = \frac{1}{j}\frac{d}{d\omega}(\Phi_X(\omega))|_{\omega=0} = \frac{1}{j}\frac{d}{d\omega}(\cos^N \omega)|_{\omega=0} = 0$$

(b)

$$E(X^2) = \frac{1}{j^2}\frac{d^2}{d\omega^2}(\Phi_X(\omega))|_{\omega=0} = \frac{1}{j^2}\frac{d^2}{d\omega^2}(\cos^N \omega)|_{\omega=0} = N$$

In case $N = 2, X = Y_1 + Y_2, E({Y_k}^2) = 1, E(Y_k) = 0$;

$$\begin{aligned}E(X^2) &= E((Y_1 + Y_2)^2) \\ &= E({Y_1}^2) + E({Y_2}^2) + 2E(Y_1 Y_2) \\ &= E({Y_1}^2) + E({Y_2}^2) + 2E(Y_1)E(Y_2) \\ &= 2 = N\end{aligned}$$

Verified!

(d) Using the Chebychev Inequality, we obtain:

$$P(|X - EX| > N) \leq \frac{\sigma^2}{N^2},$$

with $\sigma^2 = E(X^2) - E(X)^2 = N, EX = 0$.

This yields

$$P(|X| > N) \leq \frac{N}{N^2} = \frac{1}{N}.$$