ECSE 304-305B   Assignment 6 Solutions   Winter 2007

Question 6.1

A positive scalar random variable X with a density is such that $EX = \mu < \infty$, $EX^2 = \infty$.

(a) Using the Markov or Chebyshev inequalities, estimate $P(X^2 \geq \alpha^2)$, $\alpha > 0$.

(b) Explain why the distribution function $F_X(x) = P(X \leq x)$, $x \in \mathbb{R}$ is a continuous function of x.

(c) Show that there exist a real number $\gamma > 0$ such that $e^{-\gamma} = P(X \leq \gamma)$

(d) Split an integral representing Ee^{-X} at $\gamma > 0$, where γ is given in (c), and hence (justifying each step) give a lower bound for Ee^{-X} of the form $ae^{-2\gamma}$, $a > 0$. Determine a value for the positive number a.

Q6.1 Solution:

(a) Since X is a positive scalar R.V., and $\alpha > 0$, $P(X^2 \geq \alpha^2) = P(X \geq \alpha)$. Using Markov's Inequality, we get:

$$P(X^2 \geq \alpha^2) \leq \frac{\mu}{\alpha}$$

(b) Since the density for R.V. X exists, the distribution function $F_X(x) = P(X \leq x), x \in \mathbb{R}$ is continuous since the integral of a function must be continuous (when it is finite).

(c) Consider the function $H(x) = F_X(x) - e^{-x}, x \geq 0$. $H(x)$ is continuous and $H(0) = -1$ and $lim_{x \to \infty} H(x) = 1$.

Therefore, $\exists\gamma > 0, H(\gamma) = 0$, i.e. $P(X \leq \gamma) = e^{-\gamma}$

(d)

$$
\begin{aligned}
Ee^{-X} &= \int_0^\infty e^{-x} f_X(x)dx \\
&= \int_0^\gamma e^{-x} f_X(x)dx + \int_\gamma^\infty e^{-x} f_X(x)dx \\
&\geq \int_0^\gamma e^{-x} f_X(x)dx, \text{ since } e^{-x} \geq 0 \text{ and } f_X(x) \geq 0 \\
&\geq e^{-\gamma} \int_0^\gamma f_X(x)dx, \text{ since } e^{-x} \geq e^{-\gamma} \text{ for } x \in (0, \gamma) \\
&= e^{-\gamma} P(X \leq \gamma) \\
&= e^{-2\gamma} = ae^{-2\gamma}, a = 1
\end{aligned}
$$

Question 6.2

Find (a) the mean value μ, and (b) the variance σ^2 of an RV X with the *Laplace density*

$$f_X(x) = \frac{1}{2b} e^{-2|x-m|/2b},$$

where b and m are real constants, $b > 0$ and $-\infty < m < \infty$.

Find the corresponding characteristic function $\Phi_X(\omega)$ and verify the values found above for μ, σ^2 by use of the Moment Theorem.

Q6.2 Solution: We can find the mean and variance of X from a table or by integrating.

$$
\begin{aligned}
EX &= \int_{-\infty}^{\infty} (x - m + m) f_X(x) dx \\
&= \int_{-\infty}^{\infty} (x - m) \frac{1}{2b} e^{-2|x-m|/2b} dx + \int_{-\infty}^{\infty} m f_X(x) dx \\
&= \int_{-\infty}^{\infty} y \frac{1}{2b} e^{-2|y|/2b} dy + m \\
&= 0 + m \quad \text{(by symmetry of the integrand).}
\end{aligned}
$$

Integration by parts can be used to obtain EX^2. In the end, we find:

$$
\mu = m,
$$

$$
\sigma^2 = 2b^2.
$$

The characteristic function is given by:

$$
\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) dx = \int_{-\infty}^{\infty} e^{j\omega x} \tfrac{1}{2b} e^{-2|x-m|/2b} dx.
$$

Using the hint, we set the parameter m=0, making X a zero-mean R.V. The characteristic function in this case is given by:

$$
\begin{aligned}
\Phi_X(\omega) &= \int_{-\infty}^{\infty} e^{j\omega x} \frac{1}{2b} e^{-2|x|/2b} dx \\
&= \frac{1}{2b} \int_{-\infty}^{0} e^{x(\frac{1}{b} + j\omega)} dx + \frac{1}{2b} \int_{0}^{\infty} e^{x(-\frac{1}{b} + j\omega)} dx \\
&= \frac{1}{2b} \left(\frac{1}{\frac{1}{b} + j\omega} e^{x(\frac{1}{b} + j\omega)} |_{-\infty}^{0} - \frac{1}{-\frac{1}{b} + j\omega} e^{x(-\frac{1}{b} + j\omega)} |_{0}^{\infty} \right) \\
&= \frac{1}{2b} \left(\frac{1}{\frac{1}{b} + j\omega} + \frac{1}{\frac{1}{b} - j\omega} \right) \\
&= \frac{1}{1 + b^2 \omega^2}
\end{aligned}
$$

Now for a R.V. X of mean m, and using the formula on page 8 of the Lecture to account for the shift, the characteristic function becomes:

$$\Phi_X(\omega) = \frac{e^{jm\omega}}{1 + b^2\omega^2}.$$

By the moment theorem,

$$\begin{aligned} EX &= (1/j)\frac{d}{d\omega}\frac{e^{jm\omega}}{1+b^2\omega^2}\Big|_{\omega=0} \\ &= (1/j)\left(\frac{-2b^2\omega}{(1+b^2\omega^2)^2}e^{jm\omega} + \frac{jme^{jm\omega}}{1+b^2\omega^2}\right)\Big|_{\omega=0} \\ &= m. \end{aligned}$$

By taking one more derivative, we find

$$\begin{aligned} EX^2 &= -\frac{d}{d\omega}\left(\frac{-2b^2\omega}{(1+b^2\omega^2)^2}e^{jm\omega} + \frac{jme^{jm\omega}}{1+b^2\omega^2}\right)\Big|_{\omega=0} \\ &= -\Big(\frac{-(2b^2)(1+b^2\omega^2)^2 + 2b^2\omega(1+b^2\omega^2)(2b^2\omega)}{(1+b^2\omega^2)^4}e^{jm\omega} + \frac{-2b^2\omega}{(1+b^2\omega^2)^2}(jm)e^{jm\omega} \\ &\quad + \frac{-2b^2\omega}{(1+b^2\omega^2)^2}(jm)e^{jm\omega} + \frac{1}{(1+b^2\omega^2)^2}(jm)^2e^{jm\omega}\Big)\Big|_{\omega=0} \\ &= 2b^2 + m^2. \end{aligned}$$

Hence, the variance is $2b^2$.

Question 6.3

(a) The exponential random variable Z has the density $f_Z(\cdot) = \{5\mu e^{-5\mu z}, z \in R_+, \mu > 0\}$. Find the characteristic function $\Phi_Z(\omega), \omega \in R$.

(b) Let the random variable W be defined by $W = 3(Z_1) - 3(Z_2)$, where Z_1 and Z_2 are independent identically distributed exponential random variables with parameter λ. Find $\Phi_W(\omega), \omega \in R$.

(c) Using the one-to-one relation of characteristic functions and densities and part (b), find the density of W. (Hint: check Question 2.)

Q6.3 Solution:

(a) We can obtain the characteristic function from a table or by integration. Let's integrate.

$$\Phi_Z(w) = \int_0^\infty e^{jtx} 5\mu e^{-5\mu x} dx = \frac{5\mu}{5\mu - jw}.$$

(b) First, observe that the characteristic function of $-3Z_2$ is

$$\Phi_{-3Z_2}(w) = \Phi_{Z_2}(-3w),$$

by the scaling property seen earlier. By independence, we have

$$\begin{aligned}\Phi_W(w) &= \Phi_{3Z_1}(w)\Phi_{-3Z_2}(w) \\ &= \Phi_{Z_1}(3w)\Phi_{Z_2}(-3w) \\ &= \frac{\lambda}{\lambda - j3w}\frac{\lambda}{\lambda + j3w} \\ &= \frac{\lambda^2}{\lambda^2 + 9w^2} = \Phi_X(w).\end{aligned}$$

(c) Since $\Phi_W(t) = \Phi_X(t)$, by the inversion formula, X and W have the same distribution if we let $\lambda = \frac{3}{b}$ and $m = 0$. Hence, for all x

$$f_W(x) = \frac{\lambda e^{-\frac{\lambda|x|}{3}}}{6}$$

.

Question 6.4

Many people believe that the daily change of price of a company's stock on the stock market is a random variable with mean 0 and variance σ^2. That is, if Y_n represents the price of the stock on the nth day, then

$$Y_n = Y_{n-1} + X_n \qquad n \geq 1$$

where X_1, X_2,... are independent and identically distributed random variables with mean 0 and variance σ^2. Suppose that the stock's price today is 100. If $\sigma^2 = 1$, what can you say about the probability that the stock's price will exceed 105 after 10 days?

Q6.4 Solution:

We have $Y_{10} = Y_0 + \sum_{i=1}^{10} X_i$, $E(Y_{10}) = \mu = 100$ and $E((Y_{10} - \mu)^2) = \sigma^2 = 10$. We wish to apply Chebyshev's inequality

$$P(|Y_{10} - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Noting the symmetry of Y_{10}, we can half Chebyshev's inequality, drop the absolute value and hence obtain a bound for $P(Y_{10} > 105)$ (with $k = \frac{5}{\sqrt{10}}$)

$$P(Y_{10} - 100 \geq 5) \leq \frac{1}{2k^2} = \frac{1}{5}$$