ECSE 304-305B Solutions to Assignment 5 Winter 2007

Question 5.1 (SG p 174)

(i) Show that $\Sigma_{k=1}^{\infty} k\rho^k = \frac{\rho}{(1-\rho)^2}, \quad 0 < \rho < 1$

(ii) A couple decides to continue having children until they have one of each sex. Assume the events of having a boy or a girl at different births are independent, and at any birth it is equiprobable that they have a girl or a boy. How many children should this couple expect?

Solution:

(i) It is known that

$$\sum_{k=1}^{\infty} \rho^k = \frac{\rho}{1-\rho}.$$

Differentiate both sides, and then we obtain

$$\sum_{k=1}^{\infty} k\rho^{k-1} = \frac{1}{(1-\rho)^2},$$

that is to say

$$\sum_{k=1}^{\infty} k\rho^k = \rho \sum_{k=1}^{\infty} k\rho^{k-1} = \frac{\rho}{(1-\rho)^2}.$$

(ii) Define the random variable X as the number of children the couple has until they have one of each sex. We can check that

$$\begin{aligned} P(X = n) &= P(\text{n-1 boys and the nth child is a girl}) + P(\text{n-1 girls and the nth child is a boy}) \\ &= (\frac{1}{2})^{n-1}\frac{1}{2} + (\frac{1}{2})^{n-1}\frac{1}{2} \\ &= (\frac{1}{2})^{n-1}, \qquad 2 \leq n < \infty \end{aligned}$$

since the events of having a boy or girl at different births are independent.

So the expectation of the number of the children could be calculated as follows:

$$\begin{aligned} E(\text{the number of the children}) &= E(X) \\ &= \sum_{n=2}^{\infty} nP(X=n) \\ &= \sum_{n=2}^{\infty} n(\frac{1}{2})^{n-1} \\ &= \sum_{n=1}^{\infty} n(\frac{1}{2})^{n} + \sum_{n=1}^{\infty} (\frac{1}{2})^{n} \\ &= 3 \end{aligned}$$

□

Question 5.2 (SG p 182)

What are the expected number, the variance and the standard deviation (= the square root of the variance) of the number of spades in a poker hand? (A poker hand is a set of five cards that are randomly selected (i.e. the EPP applies) from an ordinary deck of 52 cards.) Give your answer to three decimal places.

Solution:

Define a random variable X as the number of spades in a poker hand. And we know

$$\begin{aligned} P(X=n) &= P(n \text{ are spades and } 5-n \text{ are not spades}) \\ &= \frac{C_{13}^{n} C_{39}^{5-n}}{C_{52}^{5}}, \qquad 0 \le n \le 5. \end{aligned}$$

So,

$$\begin{aligned} E(X) &= \sum_{n=0}^{5} nP(X=n) \\ &= \sum_{n=0}^{5} n \frac{C_{13}^{n} C_{39}^{5-n}}{C_{52}^{5}} \\ &\approx 1.248, \\ Var(X) &= \sum_{n=0}^{5} [n - E(X)]^2 P(X=n) \approx 0.866, \\ \sigma &= \sqrt{Var(X)} \approx 0.931. \end{aligned}$$

□

Question 5.3

Let X be a continuous random variable (i.e. a not a discrete random random variable, hence it takes an uncountable set of values) whose probability distribution function has the density

$$f(x) = 6x(1 - x), \qquad 0 < x < 1. \tag{1}$$

What is the probability that X takes a value within two standard deviations of the mean? (That is to say, what is the probability that X is less than or equal to the mean plus two standard deviations but greater than or equal to the mean minus two standard deviations?)

Solution

We have

$$\begin{aligned} E(X) &= \int_0^1 xf(x)dx = \frac{1}{2} \\ Var(X) &= \int_0^1 (x - E(x))^2 f(x)dx = \frac{1}{20}, \\ \sigma &= \sqrt{Var(X)} = \frac{1}{\sqrt{20}}. \end{aligned}$$

So,

$$P[E(X) - 2\sigma \le X \le E(X) + 2\sigma] = \int_{E(X)-2\sigma}^{E(X)+2\sigma} f(x)dx \approx 0.984.$$

□

Question 5.4

Each of an i.i.d. sequence of random variables $X = \{X_n; n \in \mathbf{Z}_1\}$, where $\mathbf{Z}_1 = \{1, 2, \ldots\}$, has the probability density $(2\pi 9)^{-1/2}\exp(-\frac{x^2}{18} + \frac{x}{3} - \frac{1}{2})$, $x \in I\!R$.

(i) Find the mean $\mu = EX_n$, and the variance σ^2 of $X_n; n \in \mathbf{Z}_1$. (Hint: it is not necessary to use integration, just use the standard form of the Gaussian density.)

(ii) Use Chebychev's inequality to find an upper bound on the probability that any one of these random variables takes a value greater than or equal to 3 units away from its mean.

Solution

(i) The probability density of Gaussian distribution is equal to

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

Here we have

$$f(x) = \frac{1}{\sqrt{2\pi 9}} \exp\left\{-\frac{(x-3)^2}{2\cdot 9}\right\}.$$

So $\mu = 3$ and $\sigma^2 = 9$

(ii)Using Chebychev's inequality,

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2},$$

when $k = 3$, the desired upper bound on the probability that anyone of these i.i.d. random variables takes a value greater than or equal to 3 units away from its mean is

$$1 - P[(X_n - \mu) < 3]^n = 1 - \{1 - P[(X_n - \mu) \geq 3]\}^n \leq 1 - (\frac{1}{2})^n,$$

since for a random variable X with Gaussian distribution

$$\begin{aligned} P[(X-\mu) \geq 3] &= P[(X-\mu) \leq -3], \text{ and} \\ P[(X_n-\mu) \geq 3] &\leq \frac{1}{2}. \end{aligned}$$

Solution if Q4(ii) had used 3σ instead of 3 units.

Using Chebychev's inequality,

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2},$$

when $k = 3$, the desired upper bound on the probability that anyone of these i.i.d. random variables takes a value greater than or equal to 3σ away from its mean is

$$1 - P[(X_n - \mu) < 3\sigma]^n = 1 - \{1 - P[(X_n - \mu) \geq 3\sigma]\}^n \leq 1 - (\frac{17}{18})^n,$$

since for a random variable X with Gaussian distribution

$$\begin{aligned} P[(X-\mu) \geq 3\sigma] &= P[(X-\mu) \leq -3\sigma], \textit{ and} \\ P[(X_n-\mu) \geq 3\sigma] &\leq \frac{1}{2\cdot 9^2} = \frac{1}{18}. \end{aligned}$$

□