**ECSE 304-305B Assignment 4 Solution Winter 2007**

**Question 4.1 (SG p. 152)**

Let $X$ be a random point be selected from the interval $0, 3)$ (i.e. by the EPP, or, equivalently, such that $X$ has a linearly increasing distribution function). What is the probability that $X^2-5X+6 > 0$?

**Question 4.1 (Solution)** We have that

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{3} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

and since

$$P(X^2 - 5X + 6 > 0) = 1 - P(2 \leq X \leq 3)$$

we can compute $P(X^2 - 5X + 6 > 0) = 1 - (F_X(3) - F_X(2)) = \frac{2}{3}$

**Question 4.2**

For constants $a$ and $b$, a random variable $X$ has PDF

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 5, \\ 0 & otherwise \end{cases}$$

What conditions on $a$ and $b$ are necessary and sufficient to guarantee that $f_X(x)$ is a valid PDF?

**Question 4.2 (Solution)**

For $f_X(x)$ to be a valid PDF, it needs to satisfy two conditions:

1.$f_X(x) \geq 0$ for $x \in [0, 5]$.

2.$\int_0^5 f_X(x)dx = 1$.

From condition (2) we get,

$1 = \int_0^5 (ax^2 + bx)dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_{x=5} - \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_{x=0}$
$= \frac{125}{3}a + \frac{25}{2}b$

Thus, $a = \frac{3}{125}\left(1 - \frac{25}{2}b\right)$

To satisfy condition (1), we need, $ax^2 + bx \geq 0$ for $x \in [0, 5]$

Thus, $\frac{3}{125}\left(1 - \frac{25}{2}b\right)x^2 + bx \geq 0$ for $x \in [0, 5]$

We notice that the quadratic equation $\frac{3}{125}\left(1 - \frac{25}{2}b\right)x^2 + bx = 0$ has two roots, $x_1 = 0$ and $x_2 = \frac{-125b}{3\left(1 - \frac{25}{2}b\right)}$, except when $b = \frac{2}{25}$, then the only root is $x_1 = 0$.

First we consider the case where the coefficient of $x^2$ is positive, i.e. $b < \frac{2}{25}$. In that case, $f_X(x)$ is always negative between the roots and positive outside the roots. Thus, condition (1) is satisfied only if $x_2 \leq 0$, i.e. $b \geq 0$.

Next, we consider the case where the coeficient of $x^2$ is negative, i.e. $b > \frac{2}{25}$. In this case $f_X(x)$ is always positive between the roots, and so we need $x_2 \geq 5$ to satisfy condition (1).

$\frac{-125b}{3\left(1 - \frac{25}{2}b\right)} \geq 5 \Rightarrow b \leq \frac{6}{25}$.

Finally, if $b = \frac{2}{25}$, then $f_X(x) = \frac{2}{25}x \geq 0$ for $x \in [0, 5]$, and condition (1) is satisfied.

Thus, for $f_X(x)$ to be a valid PDF, we need:

$$\begin{cases} 0 \leq b \leq \frac{6}{25} \\ a = \frac{3}{125}\left(1 - \frac{25}{2}b\right) \end{cases}$$

**Question 4.3**

A student is allowed to take a certain test three times and the student's final score will be the maximum of the test score. Thus

$$\mathbf{X} = max\{\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}\}, \tag{1}$$

where $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}$ are the three test scores and $\mathbf{X}$ is the final score. Assume the score takes values $x, 0 \leq x \leq 100$, and the results of the tests form an independent set of random variables, each with distribution function $F_{X_i} = F(x) = P(\mathbf{X_i} \leq x), \;\; 1 \leq i \leq 3$. Let the distribution function of $\mathbf{X}$ be written as

$$F_{\mathbf{X}}(x) = \{P(\mathbf{X} \leq x), \quad 0 \leq x \leq 160\}. \tag{2}$$

(a) Find which of the following gives $F_{\mathbf{X}}(\cdot)$ and give the derivation:

(i) $F^{1/3}(x^3)$,  (ii) $F^3(x)$,  (iii) $F(x^3)$

(b) If $F(x) = \{0, x < 0;\ \frac{x}{160},\ 0 \leq x \leq 160;\ 1, 160 \leq x\}$, and given that any score $\mathbf{X}_i$ is uniformly distributed over $[0, 160]$, find $F_{\mathbf{X}}(120) = Pr\{\mathbf{X} \leq 120\}$.

**Question 4.3 (Solution)**

(a) We have that $\mathbf{X} = max\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. Noting that the event $\{x \leq \mathbf{X}\}$ is equivalent to the event $\{\mathbf{X}_1 \leq x\} \cap \{\mathbf{X}_2 \leq x\} \cap \{\mathbf{X}_3 \leq x\}$ we have that

$$
\begin{aligned}
F_{\mathbf{x}}(x) &= \{P(\mathbf{X} \leq x), 0 \leq x \leq 160\} \\
&= P(\max\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\} \leq x) \\
&= P(\{\mathbf{X}_1 \leq x\} \cap \{\mathbf{X}_2 \leq x\} \cap \{\mathbf{X}_3 \leq x\}) \\
&= \prod_{i=1}^{3} P(\mathbf{X}_i \leq x), \quad \text{since } \mathbf{X}_i\text{'s are independent} \\
&= F^3(x).
\end{aligned}
$$

(b) $F_{\mathbf{X}}(120) = P(\mathbf{X} \leq 120) = F^3(120) = \left(\frac{120}{160}\right)^3 = \frac{27}{64}$