ECSE 304-305B Solutions to Assignment 3 Winter 2007

3.1 Solution:

(i) (a)

$$
\begin{aligned}
P(NF|MA) &= P(NF|CG, MA)P(CG|MA) + P(NF|FK, MA)P(FK|MA) \\
&= P(NF|CG)P(CG|MA) + P(NF|FK)P(FK|MA) \\
&\qquad \text{(by Markov Property)} \\
&= \frac{1}{2} \cdot [P(CG|MA, SG)P(SG|MA)] \\
&\quad + \frac{1}{2} \cdot [P(FK|MA, CA)P(CA|MA) + P(FK|SG, MA)P(SG|MA)] \\
&\qquad \text{(by Markov Property)} \\
&= \frac{1}{2} \cdot \left[\frac{1}{2} \cdot \frac{1}{3}\right] + \frac{1}{2} \cdot \left[1 \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3}\right] \\
&= \frac{1}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(CS|MA) &= P(CS|CG, MA)P(CG|MA) + P(CS|FK, MA)P(FK|MA) \\
&= P(CS|CG)P(CG|MA) + P(CS|FK)P(FK|MA) \\
&\qquad \text{(by Markov Property)} \\
&= \frac{1}{2} \cdot [P(CG|SG, MA)P(SG|MA)] \\
&\quad + \frac{1}{2} \cdot [P(FK|CA, MA)P(CA|MA) + P(FK|SG, MA)P(SG|MA)] \\
&\qquad \text{(by Markov Property)} \\
&= \frac{1}{2} \cdot \left[\frac{1}{2} \cdot \frac{1}{3}\right] + \frac{1}{2} \cdot \left[1 \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3}\right] \\
&= \frac{1}{2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(CS|A) &= P(CS|MA, A)P(MA|A) \\
&= P(CS|MA) \cdot \frac{1}{2} = \frac{1}{4}
\end{aligned}
$$

(ii)(a)

$$\begin{aligned}
P(MA|CS) &= \frac{P(MA \cap CS)}{P(CS)} = \frac{P(CS|MA)P(MA)}{P(CS)} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{4}} = 1
\end{aligned}$$

(b)

$$P(CA|SEC) = \frac{P(CA \cap SEC)}{P(SEC)} = \frac{P(SEC|CA)P(CA)}{P(SEC)}$$

where

$$\begin{aligned}
P(CA) &= P(CA|MA)P(MA|A)P(A) = \frac{1}{3} \\
P(SEC|CA) &= P(SEC|NF)P(NF|FK)P(FK|CA) + P(SEC|CS)P(CS|FK)P(FK|CA) \\
&= \frac{5}{6} \\
P(SG) &= P(SG|MA)P(MA|A)P(A) = \frac{1}{6} \\
P(SEC|SG) &= P(SEC|CS)P(CS|CG)P(CG|SG) + P(SEC|NF)P(NF|CG)P(CG|SG) \\
&\quad + P(SEC|CS)P(CS|FK)P(FK|SG) + P(SEC|NF)P(NF|FK)P(FK|SG) \\
&= \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \\
&= \frac{5}{6}
\end{aligned}$$

and

$$P(SEC) = P(SEC|CA)P(CA) + P(SEC|SG)P(SG) = \frac{5}{12}.$$

Hence, we have

$$P(CA|SEC) = \frac{\frac{5}{6} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{5}{6}} = \frac{2}{3}.$$

(iii) The transition matrix $\mathbf{T}$ is given as

$$\mathbf{T} := \begin{bmatrix} 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and hence the probability distribution on day 3 is given as

$$\begin{aligned} \mathbf{P}_3 &= \mathbf{T}^2 \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \end{aligned}$$

i.e, $P(CS) = P(NF) = \frac{1}{2}$.

3.2 Solution:

(i)

$$P(n|\alpha) = P(n\delta|\alpha) + P(n\gamma|\alpha) = P(n|\delta)P(\delta|\alpha) + P(n|\gamma)P(\gamma|\alpha) = \frac{3}{16} + \frac{1}{4} = \frac{7}{16}$$

(ii)

$$P(n|\beta) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

(iii)

$$P(\alpha|n) = \frac{P(\alpha n)}{P(n)} = \frac{P(n|\alpha)P(\alpha)}{P(n|\alpha)P(\alpha) + P(n|\beta)P(\beta)} = \frac{\frac{7}{16}}{\frac{7}{16} + \frac{5}{8}} = \frac{7}{17}$$

3.3 Solution

(a) P(Stage 1: n steps; Stage 2: m steps)=$(1-p)^{n-1}p \cdot (1-q)^{m-1}q$.

(b) Define event E as the specified event in (b); then

$P(E) = (\Sigma_{n=1}^{\infty}(1-p)^{n-1}p) \cdot (1-q)^{m-1}q = (1-q)^{m-1}q$.

(c) Define event A as the specified event in (c); then

$P(A) = (\Sigma_{n=0}^{\infty}(1-p)^{2n+1-1}p) \cdot 1 = \frac{2}{3}$.

3.4 Solution

(a) The transition matrix T is:

$$T = \begin{bmatrix} \alpha & \alpha' & p \\ \beta & \beta' & 1-p \\ 1-\alpha-\beta & 1-\alpha'-\beta' & 0 \end{bmatrix}.$$

(b) In the case specified in part (b),

$$T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

After 2 jumps the occupation probability for a process which starts in state S_1 with probability 1 is:

$$T^2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}.$$

Hence, we have $P_L = \frac{1}{4}$.

(c)(i) For $(P_L, P_R, P_1) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$:

$$T \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

(ii) For $(P_L, P_R, P_1) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$:

$$T \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{bmatrix}.$$

Hence, we have only case (ii): $(P_L, P_R, P_1) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as a steady state probability for the markov chain.