ECSE 304-305B Solution to Assignment 1 Winter 2007

Question 1.1

Solution:

$$\begin{aligned}
P(F_2 \cap F_3) &= P((F_2 \cap F_3 \cap F_1) \cup (F_2 \cap F_3 \cap F_1^c)) \\
&= 0.003 + 0.019 = 0.022 \\
P(F_1^c) &= P((F_1^c \cap (F_2 \cap F_3)) \cup (F_1^c \cap (F_2 \cap F_3)^c)) \\
&= P(F_1^c \cap (F_2 \cap F_3)) + P(F_1^c \cap (F_2 \cap F_3)^c)) \\
&= 0.019 + 0.009 = 0.028 \\
P(F_1) &= 1 - P(F_1^c) = 1 - 0.028 = 0.972
\end{aligned}$$

Since all information given about F_2 or F_3 appears in probabilities involving $(F_2 \cap F_3), (F_2 \cap F_3)^c, F_1, F_1^c$, it is not possible to separately compute $P(F_2)$ or $P(F_3)$.

Question 1.2

Solution:

(i) $\alpha = \gamma$. Because $X \cap (\cup_{i=1}^N A_i) = \cup_{i=1}^N (X \cap A_i)$, and A_i are disjoint $\forall i$, we have $P(X \cap (\cup_{i=1}^N A_i)) = P(\cup_{i=1}^N (X \cap A_i)) = \sum_{i=1}^N P(X \bigcap A_i)$.

For α and β: For example, let $N = 2$, $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, $A_1 \cap A_2 = \emptyset$, and $P(A_1 \cup A_2) = 0.5$, then $\alpha = P(A_2 \cap (A_1 \cup A_2)) = P(A_2) > 0$, and $\beta = P(X)P(A_1 \cup A_2) = P(A_2)/2 > 0$, hence, $\alpha \neq \beta$.

(ii) Since $\{A_i, 1 \leq i < \infty\}$ and $\{B_i, 1 \leq i < \infty\}$ are pairwise disjoint, the sets $\{A_i \cap B_i, 1 \leq i < \infty\}$ are pairwise disjoint. Hence, $P(\cup_{i=1}^\infty (A_i \cap B_i)) = \sum_{i=1}^\infty P(A_i \cap B_i)$.

(iii) If $A \cap B = \emptyset$, then $A_i \cap B_i = \emptyset$, and $\cup_{i=1}^\infty (A_i \cap B_i) = \emptyset$, i.e. $P(\cup_{i=1}^\infty (A_i \cap B_i)) = 0$.

Question 1.3

Solution:

$S = \{H, TH, T^2H, ..., T^{N-1}H, ...\}$ (a) The events $H, TH, T^2H, ..., T^{N-1}H, ... \subset S$ are disjoint since T^jH occurs only if T^iH, $i < j$, does not occur and then T^kH, $k > j$ cannot occur.

(b) BE(Branding Experiment) stops at the outcome of n-th toss if and only if $T^{n-1}H$ occurs, where $P(T^{n-1}H) = P(T)^{n-1}P(H) = (\frac{2}{3})^{n-1}\frac{1}{3}$ (by independence) Hence BE stops at the outcome of the n-th toss, $n \leq N$, if $E^N = H \cup TH \cup ... \cup T^{N-1}H$ occurs, By axiom 3, E^N has probability $P(E^N) = \frac{1}{3} + (\frac{2}{3})\frac{1}{3} + ... + (\frac{2}{3})^{N-1}\frac{1}{3} = \frac{1-(\frac{2}{3})^N}{1-\frac{2}{3}} = 1 - (\frac{2}{3})^N$

(c) The events $\{H, TH, T^2H, ..., T^{N-1}H, ...\}$ are not pairwise independent since they are disjoint, or events T^iH, T^jH, $0 \leq i < j < \infty$ are not independent since $0 = P(T^iH \cap T^jH) \neq P(T^iH)P(T^jH) = (\frac{2}{3})^{i-1}\frac{1}{3}(\frac{2}{3})^{j-1}\frac{1}{3}$, $0 \leq i < j < \infty$.

Question 1.4

Solution:

(a) The sample space for this experiment where the outcome is taken to be pair (T_1, T_2) is shown in Fig.1.

(b) The region A of the plane corresponding to the event "student is awake at 9pm" is shown in Fig.2.

(c) The set B in the plane corresponding to the event "student is asleep more time than the student is awake" is shown in Fig.3.

(d) The region corresponding to the event $D = A^c \cap B$, which means "student is not awake at 9pm and the student is asleep more time than being awake" is shown in Fig.4.

The probability of the event D is

$$P_D = \frac{\frac{1}{2} \times 15^2 - \frac{1}{2} \times 3^2 + \frac{1}{2} \times 9^2}{\frac{1}{2} \times 24^2} \approx 0.516$$

Question 1.5

Solution:

(a)

$$P(\text{exactly 4 numbers correct}) = \frac{C_4^6 \times C_{6-4}^{49-6}}{C_6^{49}} = 9.6862 \times 10^{-4}$$

(b)

$$\begin{aligned} P(\text{exactly 1 or 2 numbers correct}) &= P(\text{exactly 1 number correct}) + P(\text{exactly 2 numbers correct}) \\ &= \frac{C_1^6 \times C_{6-1}^{49-6}}{C_6^{49}} + \frac{C_2^6 \times C_{6-2}^{49-6}}{C_6^{49}} \approx 0.545 \end{aligned}$$

(c) Define the events A, B as exactly one number correct on one of two independent tickets, say Ticket A and Ticket B.

$$\begin{aligned} & P(\text{one number correct on at least one of two independent tickets}) \\ = \ & P(A \cup B) \\ = \ & P(A) + P(B) - P(A \cap B) \\ = \ & P(A) + P(B) - P(A)P(B) \\ = \ & 2 \times \frac{C_1^6 \times C_{6-1}^{49-6}}{C_6^{49}} - (\frac{C_1^6 \times C_{6-1}^{49-6}}{C_6^{49}})^2 \\ \approx \ & 0.655 \end{aligned}$$

or use

$$\begin{aligned} & P(\text{one number correct on at least one of two independent tickets}) \\ = \ & 1 - P(A^c \cap B^c) \\ = \ & 1 - P(A^c)P(B^c) \\ = \ & 1 - (1 - P(A))(1 - P(B)) \\ = \ & P(A) + P(B) - P(A)P(B) \\ = \ & 2 \times \frac{C_1^6 \times C_{6-1}^{49-6}}{C_6^{49}} - (\frac{C_1^6 \times C_{6-1}^{49-6}}{C_6^{49}})^2 \\ \approx \ & 0.655 \end{aligned}$$

Question 1.6

Solution:

Define the events W_I, W_{II}, and W_{III} as the candidate winning districts I, II and III separately; we seek the probability $P(W_I \cap W_{II} \cap W_{III})$. It is known that $P(W_I \cap W_{III}) = 0.55$, $P(W_{II}^c \cap W_I) = 0.34$, and $P(W_{II}^c \cap W_{III}^c \cap W_I) = 0.15$.

Hence

$$\begin{aligned} P(W_I \cap W_{II} \cap W_{III}) &= P(W_I \cap W_{III}) - P(W_I \cap W_{II}^c \cap W_{III}) \\ &= P(W_I \cap W_{III}) - [P(W_I \cap W_{II}^c) - P(W_I \cap W_{II}^c \cap W_{III}^c)] \\ &= 0.55 - 0.34 + 0.15 \\ &= 0.36. \end{aligned}$$

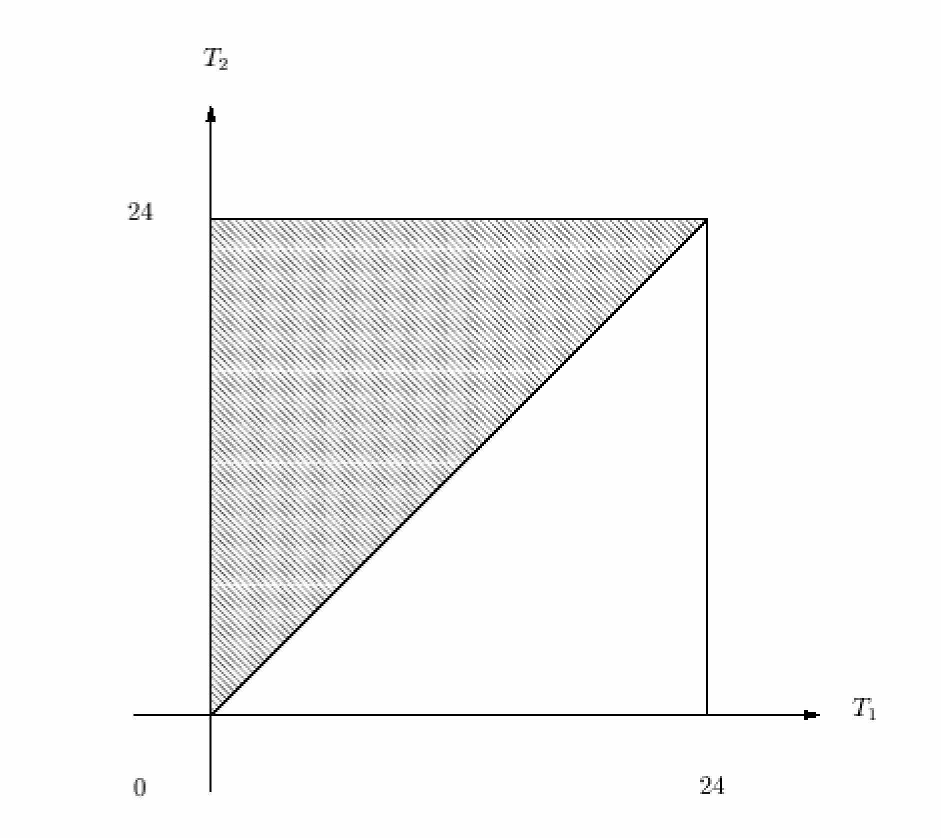

Figure 1: Question 1.4 (a): sample space S

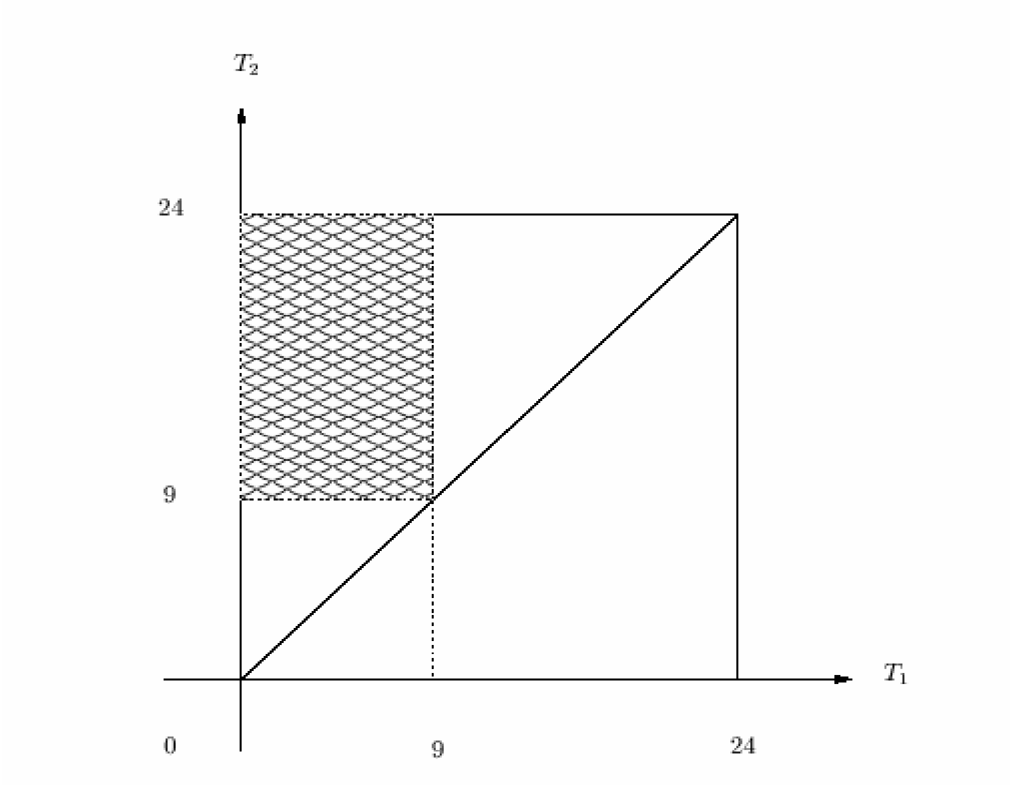

Figure 2: Question 1.4 (b): region corresponding to the event “A student is awake at 9pm”.

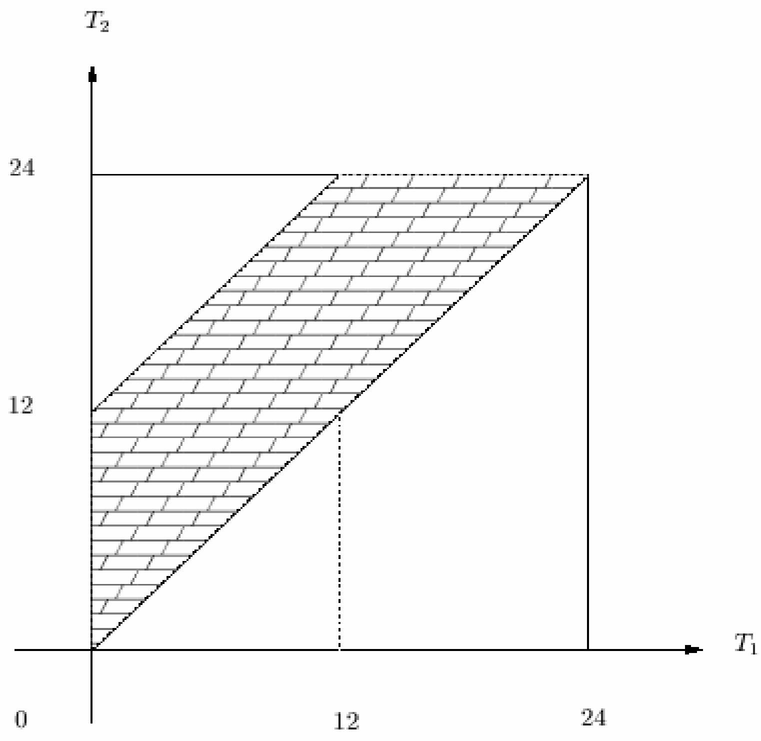

Figure 3: Question 1.4 (c): region corresponding to the event B “student is asleep more time than the student is awake”.

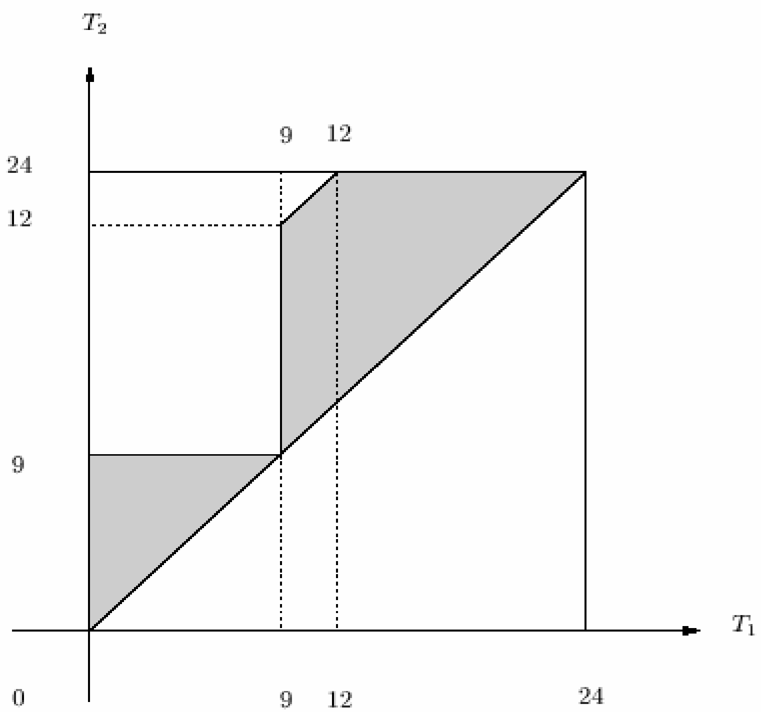

Figure 4: Question 1.4 (d): region corresponding to the event D