ECSE 304-305B   Solution to Assignment 2   Fall 2008

Question 2.1

(a) Consider the *multinomial distribution with parameters N=4, r=2* for the random labeling of objects as Black or White (which is a *Binomial distribution* with *r=2*). The probability that an element of the set of four objects will be labeled Black is *p* and White is *1-p*. Write down the formula from the Notes (webby Number 5) for the probability of the event where *m* elements are labeled Black and *4-m* are labeled White. Treating *p* as an indeterminate (i.e. an algebraic variable), expand the probabilities for all possible values of *m* and sum the resulting polynomials. Explain the answer you obtain.

(b) Let a coin have a probability $\frac{3}{4}$ of coming down Heads in a toss and $\frac{1}{4}$ Tails. Generalize the method in the example below to this biased coin case so as to find the probability of getting two or more heads in five tosses. (This is again an example of a *multinomial distribution with parameter r=2*.)

Express your answer in terms of fractions and then evaluate it to three decimal places.

Example: Find the probability of getting four or more heads in six tosses of an unbiased coin.

Solution: $P(4 \text{ or more heads in 6 tosses}) = P(4) + P(5) + P(6)$ where

$$P(k) = P(k \text{ heads in 6 tosses}) = \left(\begin{array}{c} 6 \\ k \end{array} \right) /2^6,$$

so $P(4 \text{ or more heads in 6 tosses}) = (15 + 6 + 1)/2^6$=11/32.

Solution: (a) Let B denote the number of events where elements are labeled Black, and W denote the number of events where elements are labeled White. The probability of the event where m elements are labeled Black and $4 - m$ are labeled White is:

$$P(B = m, W = 4 - m) = \frac{4!}{m!(4 - m)!} p^m (1 - p)^{4-m}.$$

The possible values of m are between 0 and 4. Treating p as an indeterminate, we expand the probabilities for each possible values of m:

$$P(B = 0) = \frac{4!}{0!4!} p^0 (1 - p)^4 = (1 - p)^4,$$

$$P(B=1) = \frac{4!}{1!3!}p^1(1-p)^3 = 4p(1-p)^3,$$
$$P(B=2) = \frac{4!}{2!2!}p^2(1-p)^2 = 6p^2(1-p)^2,$$
$$P(B=3) = \frac{4!}{3!1!}p^3(1-p)^1 = 4p^3(1-p),$$
$$P(B=4) = \frac{4!}{4!0!}p^4(1-p)^0 = p^4.$$

Summing the resulting polynomial we get:

$$p^4 + 4p(1-p)^3 + 6p^2(1-p)^2 + 4p^3(1-p) + p^4 = 1,$$

since it is a summation of all possible outcomes.

b) Let $p = \frac{3}{4}$ be the probability of heads, then, we find that

$$P(k \text{ heads in 5 tosses}) = P(k) = \left(\begin{array}{c} 5 \\ k \end{array} \right) p^k \left(1-p\right)^{5-k}$$

so

$$P(\text{ 2 or more heads in 5 tosses}) = P(2) + P(3) + P(4) + P(5) = 0.984.$$

Question 2.2

All of the following parts are for Loto 6-15 as a modification of the Loto game on pages 3 and 4, of the Notes: Counting Probabilities (webct Number 5).

a) What is the probability of getting exactly three numbers correct on one ticket showing 6 distinct numbers in a draw of 6 numbers without replacement (i.e. a draw of six distinct numbers)?

Solution:

$$\begin{aligned} P(win) &= \frac{C_3^6 \times C_{6-3}^{15-6}}{C_6^{15}} \\ &= \frac{20 \times 84}{5005} \\ &= \frac{1580}{5005} \\ &= \frac{48}{143} \end{aligned}$$

b) What is the probability of getting exactly one or two numbers correct on one ticket? Give the answer as a fraction and correct to three decimal places.

Solution:

$$
\begin{aligned}
P(win) &= \frac{C_1^6 \times C_{6-1}^{15-6}}{C_6^{15}} + \frac{C_2^6 \times C_{6-2}^{15-6}}{C_6^{15}} \\
&= \frac{2646}{5005} \\
&= \frac{378}{715} \approx 0.529
\end{aligned}
$$

c) You buy two tickets which turn out to have no numbers in common. What is the probability of getting exactly two numbers correct on each of the tickets in one draw?

Solution:

$$
\begin{aligned}
P(win) &= \frac{C_2^6 \times C_4^9}{C_6^{15}} \times \frac{C_2^4 \times C_4^5}{C_6^9} \\
&= \frac{135}{1001} \approx 0.135
\end{aligned}
$$

Question 2.3

The Partition Function of Statistical Mechanics is the ratio of an exponential to the sum of exponentials (each corresponding to an energy level) as given in the last section of the Lecture 5: Counting Probabilities of the Notes. In a specific application of the Partition Function, a substance in a vessel has $n = 2^{10}$ particles at a temperature $\mu = 1/50$ in appropriate units. There are four energy levels $e_i = i^2$; for $i = \{1, 2, 3, 4\}$

(a) Find an expression for the number of particles in each energy level in the most likely configuration [subject to the number and energy constraints (1) and (2)] by applying the formulas for each n_i in the notes.

(b) Give a formula for the energy per particle in the most likely configuration [subject to (1) and (2)] by using the formula for E/N in the Notes.

Solution:

(a) Using $\mu = 1/50$ (temperature in suitable units), $n = 2^{10} = 1024$ (particles) and $k = 4$ energy

(e_i) levels, the expression for the number of particles in each energy level will be as follows:

$$n_i = \frac{1024 \exp\{-\frac{2^i}{50}\}}{\sum_{j=1}^{k} \exp\{-\frac{2^i}{50}\}}, \tag{1}$$

Let Z be the normalizing factor in Equation 1; that is $Z = \sum_{j=1}^{k} \exp\{-\frac{2^j}{50}\}$.

Then the number of particles in $e_1 = \frac{1024*2^1}{Z}$.

The number of particles in $e_2 = \frac{1024*2^2}{Z}$.

The number of particles in $e_3 = \frac{1024*2^3}{Z}$.

The number of particles in $e_4 = \frac{1024*2^4}{Z}$.

As an extra check, you can evaluate the above expressions and see whether they satisfy the first constraint (Conservation of particles). For $Z = 3.46$, the number of particles in $e_1,\ e_2,\ e_3,\ e_4$ will be $289.7,\ 272.8,\ 246.8,\ 214.6$ respectively. Then the total number of particles will be $1023.9 \approx 1024$.

(b) Using the above values, the energy per particle in the most likely configuration and under constraints (1) and (2) will be as follows:

$$\frac{E}{N} = \frac{\sum_{j=1}^{k} 2^j \exp\{-\frac{2^j}{50}\}}{Z} = 6.8717 \tag{2}$$

Question 2.4

A total of n balls are sequentially and randomly chose, without replacement, from an urn containing r red and b blue balls $(n \leq r + b)$. Given that k of the n balls are blue, what is the conditional probability that the first ball chosen is blue?

Hint: Let B_f be the event that the first ball chosen is blue and let B_k be the event that a total of k blue balls are chosen. Then using the definition of conditional probability, obtain:

$$P(B_f|B_k) = \frac{P(B_k|B_f)P(B_f)}{P(B_k)} \tag{3}$$

Next find $P(B_k)$ by using $C_k^b,\ C_{n-k}^r$ and C_n^{b+r}. Similarly find $P(B_k|B_f)$ by using $C_{k-1}^{b-1},\ C_{n-k}^r$ and C_{n-1}^{b+r-1}.

Verify you answer by showing that for both $r = 4,\ b = 2,\ n = 3,\ k = 2$ and for $r = 5,\ b = 3,\ n = 3,\ k = 2$, the conditional probability is $2/3$.

Solution:

First, we need to calculate the total probability $P(B_k)$:

$$P(B_k) = \frac{C_k^b \times C_{n-k}^r}{C_n^{b+r}}$$

Second, we need to calculate the conditional probability $P(B_k|B_f)$:

$$P(B_k|B_f) = \frac{C_{k-1}^{b-1} \times C_{n-k}^r}{C_{n-1}^{b+r-1}}$$

Finally, the prior probability $P(B_f)$ is the just the probability of choosing only one blue ball given $b + r$ balls in the urn : $P(B_f) = \frac{b}{b+r}$. The final expression for the posterior probability $P(B_f|B_k)$ will be:

$$P(B_f|B_k) = \frac{\frac{C_{k-1}^{b-1} \times C_{n-k}^r}{C_{n-1}^{b+r-1}} \times \frac{b}{b+r}}{\frac{C_k^b \times C_{n-k}^r}{C_n^{b+r}}}$$

From this point, you can proceed in either two ways. The first one, very straight forward, can be by evaluating each of the above expressions using the provided values and you will end up with the correct probability of $2/3$. The second, will be to simplify further the expression of the posterior probability until you end up with an expression of the form k/n which immediately evaluates to the desired probability.

For illustration, we shall proceed with the second approach.

$$\begin{aligned}
P(B_f|B_k) &= \frac{\frac{C_{k-1}^{b-1} \times C_{n-k}^r}{C_{n-1}^{b+r-1}} \times \frac{b}{b+r}}{\frac{C_k^b \times C_{n-k}^r}{C_n^{b+r}}} \\
&= \frac{\frac{C_{k-1}^{b-1}}{C_{n-1}^{b+r-1}} \times \frac{b}{b+r}}{\frac{C_k^b}{C_n^{b+r}}} \\
&= \frac{C_{k-1}^{b-1}}{C_k^b} \times \frac{C_n^{b+r}}{C_{n-1}^{b+r-1}} \times \frac{b}{b+r} \\
&= \frac{k}{b} \times \frac{b+r}{n} \times \frac{b}{b+r} \\
&= \frac{k}{n}
\end{aligned}$$