ECSE 304-305B Solution to Assignment 1 Winter 2008

Question 1.1

Solution:

$$
\begin{aligned}
P(F_2 \cap F_3) &= P((F_2 \cap F_3 \cap F_1) \cup (F_2 \cap F_3 \cap F_1^c)) \\
&= 0.004 + 0.023 = 0.027 \\
P(F_1^c) &= P((F_1^c \cap (F_2 \cap F_3)) \cup (F_1^c \cap (F_2 \cap F_3)^c)) \\
&= P(F_1^c \cap (F_2 \cap F_3)) + P(F_1^c \cap (F_2 \cap F_3)^c)) \\
&= 0.023 + 0.011 = 0.034 \\
P(F_1) &= 1 - P(F_1^c) = 1 - 0.034 = 0.966
\end{aligned}
$$

Since all information given about F_2 or F_3 appears in probabilities involving $(F_2 \cap F_3), (F_2 \cap F_3)^c, F_1, F_1^c$, it is not possible to separately compute $P(F_2)$ or $P(F_3)$.

Question 1.2

Solution:

$S = \{H, TH, T^2H, ..., T^{N-1}H, ...\}$

(a)

i) The events $H, TH, T^2H, ..., T^{N-1}H, ... \subset S$ are disjoint since T^jH occurs only if T^iH, $i < j$, does not occur and then T^kH, $k > j$ cannot occur.

ii) No. Since the first union is sample space S and the second set $E(S)$ is the set of all subsets of S.

(b) BE (Branding Experiment) stops at the outcome of n-th toss if and only if $T^{n-1}H$ occurs, where $P(T^{n-1}H) = P(T)^{n-1}P(H) = (\frac{3}{4})^{n-1}\frac{1}{4}$ (by independence) Hence BE stops at the outcome of the n-th toss, $n \leq N$, if $E^N = H \cup TH \cup ... \cup T^{N-1}H$ occurs, By axiom 3, E^N has probability $P(E^N) = \frac{1}{4} + (\frac{3}{4})\frac{1}{4} + ... + (\frac{3}{4})^{N-1}\frac{1}{4} = \frac{1}{4} \times \frac{1-(\frac{3}{4})^N}{1-\frac{3}{4}} = 1 - (\frac{3}{4})^N$

(c) The events $\{H, TH, T^2H, ..., T^{N-1}H, ...\}$ are not pairwise independent since they are disjoint, or events T^iH, T^jH, $0 \leq i < j < \infty$ are not independent since $0 = P(T^iH \cap T^jH) \neq P(T^iH)P(T^jH) = (\frac{3}{4})^{i-1}\frac{1}{4}(\frac{3}{4})^{j-1}\frac{1}{4}$, $0 \leq i < j < \infty$.

(d)

i) The probability that the experiment BE stops after an odd number of tosses is:

$$\sum_{k=0}^{N} \frac{1}{4}(\frac{3}{4})^{2k} = \frac{1}{4} \times \frac{1}{1-(\frac{3}{4})^2} = \frac{4}{7}.$$

ii) Let $A = \{$The experiment BE stops after an even number of tosees$\}$
and $B = \{$The experiment BE stops after an odd number of tosees$\}$ then we have

$$P(S) = P(A) + P(B) = 1.$$

Therefore, $P(A) = 1 - P(B) = 1 - \frac{4}{7} = \frac{3}{7}$.

Question 1.3

Solution:

(a) The sample space for this experiment where the outcome is taken to be pair (T_1, T_2) is shown in Fig. 1.

(b) The region A of the plane corresponding to the event "student is awake at 9pm" is shown in Fig. 2.

(c) The set B in the plane corresponding to the event "student is asleep more time than the student is awake" is shown in Fig. 3.

(d) The region corresponding to the event $D = A^c \cap B$, which means "student is not awake at 9pm and the student is asleep more time than being awake" is shown in Fig. 4.

The probability of the event D is

$$P_D = \frac{\frac{1}{2} \times 15^2 - \frac{1}{2} \times 3^2 + \frac{1}{2} \times 9^2}{\frac{1}{2} \times 24^2} \approx 0.516$$

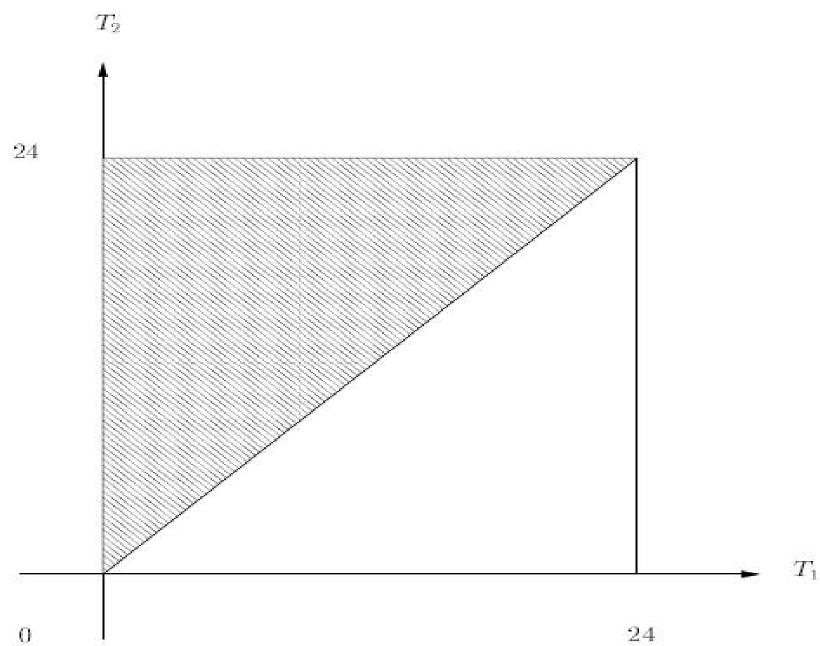

Figure 1: Question 1.3 (a): sample space S.

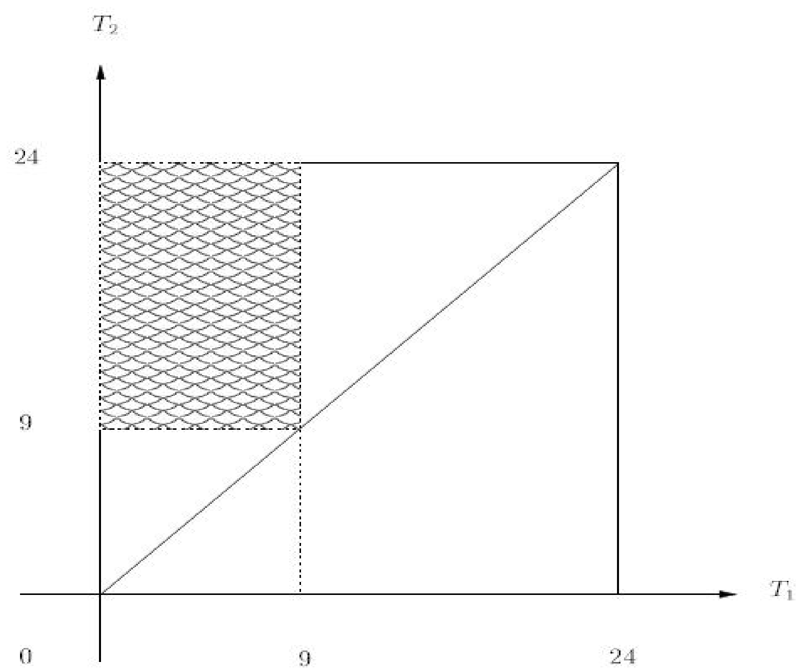

Figure 2: Question 1.3 (b): region corresponding to the event A.

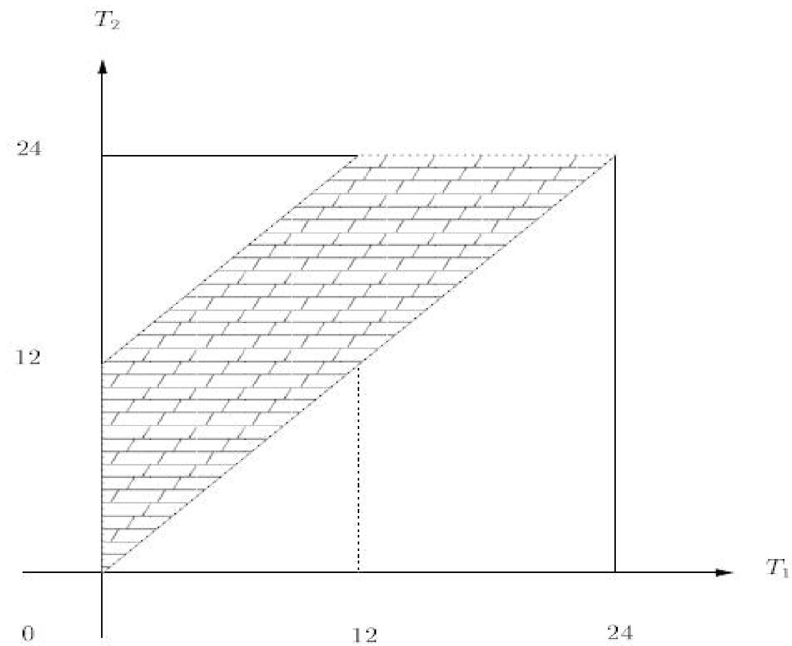

Figure 3: Question 1.3 (c): region corresponding to the event B.

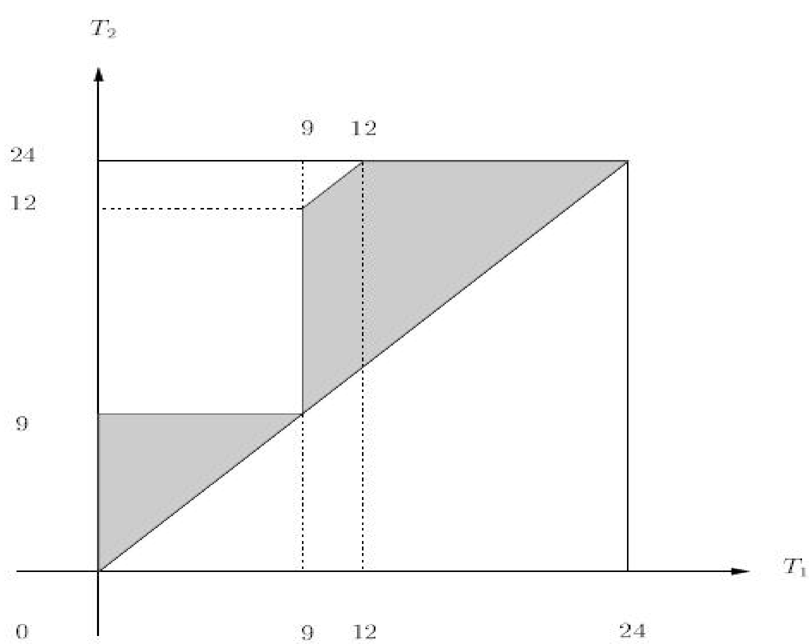

Figure 4: Question 1.3 (d): region corresponding to the event $D = A^c \cap B$.

Question 1.4

Solution:

i) $P(A^c) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Ind_k(A^c) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}[1 - Ind_k(A)] = 1 - [\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Ind_k(A)] = 1 - P(A)$.

ii) $Ind_k(A) = 0$ or 1, and $Ind_k(B) = 0$ or 1, so

$$0 \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Ind_k(A) \le 1,$$

$$0 \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Ind_k(B) \le 1.$$

Since

$$Ind_k(A) = 1 \Rightarrow Ind_k(B) = 1,$$

$$Ind_k(B) = 1 \Rightarrow Ind_k(B) = 0 \text{ or } 1,$$

then

$$\frac{1}{n}\sum_{k=1}^{n} Ind_k(A) \le \frac{1}{n}\sum_{k=1}^{n} Ind_k(B),$$

and so

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Ind_k(A) \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Ind_k(B),$$

therefore $0 \le P(A) \le P(B) \le 1$.

iii) Let $(AB)_k$, etc., denote $(A \cap B)_k$, etc., then

$$Ind_k(A \cup B) \left\{ \begin{array}{ll} 1 & \text{on } (AB)_k, \\ 1 & (AB^c)_k, \\ 1 & (A^cB)_k, \\ 0 & (A^cB^c)_k. \end{array} \right\} = Ind_k(A) \left\{ \begin{array}{ll} 1 & \text{on } (AB)_k, \\ 1 & (AB^c)_k, \\ 0 & (A^cB)_k, \\ 0 & (A^cB^c)_k. \end{array} \right\} + Ind_k(B) \left\{ \begin{array}{ll} 1 & \text{on } (AB)_k, \\ 0 & (AB^c)_k, \\ 1 & (A^cB)_k, \\ 0 & (A^cB^c)_k. \end{array} \right\}$$

$$-Ind_k(A \cap B) \left\{ \begin{array}{ll} 1 & \text{on } (AB)_k, \\ 0 & (AB^c)_k, \\ 0 & (A^cB)_k, \\ 0 & (A^cB^c)_k. \end{array} \right\}.$$

So

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} Ind_k(A\cup B) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} Ind_k(A) + \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} Ind_k(B) - \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} Ind_k(A\cap B),$$

therefore

$$P(A\cup B) = P(A) + P(B) - P(A\cap B).$$

We note that in the calculation above, $Ind_k(A\cap B) = Ind_k(A).Ind_k(B)$.

Question 1.5

Solution:

Define the events W_I, W_{II}, and W_{III} as the candidate winning districts I, II and III separately; we seek the probability $P(W_I \cap W_{II} \cap W_{III})$. It is known that $P(W_I \cap W_{III}) = 0.55$, $P(W_{II}^c \cap W_I) = 0.35$, and $P(W_{II}^c \cap W_{III}^c \cap W_I) = 0.15$.

Hence

$$\begin{aligned}
P(W_I \cap W_{II} \cap W_{III}) &= P(W_I \cap W_{III}) - P(W_I \cap W_{II}^c \cap W_{III}) \\
&= P(W_I \cap W_{III}) - [P(W_I \cap W_{II}^c) - P(W_I \cap W_{II}^c \cap W_{III}^c)] \\
&= 0.55 - 0.35 + 0.15 \\
&= 0.35.
\end{aligned}$$