## ECSE 304-305B Solutions to Assignment 8 Fall 2008

### Problem 8.1

A manufacturing system is governed by a Poisson counting process $\{N_t; 0 \leq t < \infty\}$ with rate parameter $\lambda > 0$. If a counting event occurs at an instant $t, 0 \leq t < \infty$, the system generates an item of product $P$.

The value of any product item is described by an $R$ valued Gaussian random variable $V$ with mean 5 and variance 3. (Yes, the value can be negative if the product is defective!) Each item is stored in a buffer $B$ when it is produced. $B$ is empty at $t = 0$, and the counting process and all the generated random variables $\{V_1, V_2, ...\}$ are mutually independent.

(a) The buffer is observed at an instant $t, t > 0$. What is the mean value of the number of items $N_t$ in the buffer $B$? Is this mean value finite as $t \to \infty$? If yes, what is the mean value in the limit?

(b) The total value $W_t = \Sigma_{k=1}^{N_t} V_k$ of the items in $B$ obviously has a Gaussian distribution when conditioned on the number of items $N_t$ in $B$ at an instant $t, t > 0$; find the conditional mean and variance of $W_t$.

(c) Use (b) to find the unconditional mean and variance of the total value $W_t$ of the items in $B$ at $t, t > 0$.

(d) Use the conditional and unconditional probability distributions in (b) and (c) to give a formula (involving two integrals) for the conditional probability $P(N_t = k | V_t \leq v)$ for $0 \leq k < \infty, v \in R$.

### Solution 8.1

(a) At the instant $t$, the PMF of the number $N_t$ of items in the buffer B is:

$$P(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, n = 0, 1, ...$$

Hence,

$$\begin{aligned}
E\{N_t\} &= \sum_{n=0}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{(n-1)!} \\
&= (\lambda t) e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= \lambda t, t \geq 0
\end{aligned}$$

Thus the mean goes to $\infty$ as $t \to \infty$.

(b) Conditional mean:

$$E_{|N_t}\{W_t\} = \sum_{k=1}^{N_t} E_{|N_t}\{V_k\} = 5N_t$$

Conditional variance:

$$\begin{aligned}
var_{|N_t}\{W_t\} &= E_{|N_t}\{(W_t - 5N_t)^2\} \\
&= E_{|N_t}\left(\sum_{k=1}^{N_t}(V_k - 5)\right)^2 \\
&= \sum_{k=1}^{N_t} E_{|N_t}(V_k - 5)^2 \\
&= 3N_t
\end{aligned}$$

(c) Unconditional mean of $W_t$

$$\begin{aligned}
E\{W_t\} &= E_{N_t}\{E_{|Nt}\{W_t\}\} \\
&= E_{N_t}\{5N_t\} \\
&= 5(E\{N_t\}) \\
&= 5\lambda t
\end{aligned}$$

Unconditional variance of $W_t$

$$\begin{aligned}
E\{(W_t - E\{W_t\})^2\} &= E_{N_t}\{E_{|N_t}(W_t - 3N_t)^2\} \\
&= E_{N_t}\{3N_t\} \\
&= 3\lambda t
\end{aligned}$$

(d)

$$\begin{aligned} P(N_t = k|W_t \le v) &= \frac{P(W_t \le v \cap N_t = k)}{P(W_t \le v)} \\ &= \frac{P(W_t \le v|N_t = k) \cdot P(N_t = k)}{P(W_t \le v)} \end{aligned}$$

Now define

$$\alpha_{t,k}(v) \triangleq P(W_t \le v|N_t = k) = \frac{1}{\sqrt{(2\pi)}} \frac{1}{\sqrt{3N_t}} \int_{-\infty}^{v} e^{-\frac{1}{2}\frac{(x-5N_t)^2}{3N_t}} dx$$

$$\begin{aligned} \beta_t(v) \triangleq P(W_t \le v) = \sum_{0}^{\infty} P(W_t \le v|N_t = k) \cdot P(N_t = k) &= \sum_{k=0}^{\infty} \left( e^{-\lambda t} \frac{(\lambda t)^k}{k!} \right) \int_{-\infty}^{v} \frac{e^{-\frac{1}{2}\frac{(x-5k)^2}{3k}}}{\sqrt{2\pi}\sqrt{3k}} dx \\ &= \int_{-\infty}^{v} \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} e^{-\frac{1}{2}\frac{(x-5k)^2}{3k}} dx \end{aligned}$$

Thus,

$$P(N_t = k|W_t \le v) = \frac{\alpha_{t,k}(v)}{\beta_t(v)} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

**Problem 8.2**

For the bivariate random variables $(X, Y)$ distributed as

$$f_{X,Y}(x, y) = \begin{cases} ce^{-3x}e^{-2y}, & 0 \le y \le x < \infty \\ 0, & otherwise \end{cases}$$

use the formulas for the marginal densities in the lectures to find $EX$, $EY$, and hence find the covariance function $Cov(X, Y) = E(X - EX)(Y - EY) = EXY - EXEY$ and the correlation coefficient $\rho = \frac{Cov(X,Y)}{\sqrt{\sigma_x^2}\sqrt{\sigma_y^2}}$

**Solution 8.2**

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= 1, \\
\int_{0}^{\infty}\int_{0}^{x} ce^{-3x}e^{-2y}dydx &= 1, \\
\int_{0}^{\infty} \frac{c}{2}\left(e^{-3x}-e^{-5x}\right)dx &= 1, \\
c &= 15,
\end{aligned}$$

$$\begin{aligned}
f_X(x) &= \int_0^x 15e^{-3x}e^{-2y}dy = \frac{15}{2}\left(e^{-3x}-e^{-5x}\right), \quad 0 \le x < \infty \\
f_Y(y) &= \int_0^{\infty} 15e^{-3x}e^{-2y}dx = 5e^{-5y}, \quad 0 \le y < \infty.
\end{aligned}$$

Hence, by using integration by parts, we have

$$\begin{aligned}
EX &= \int_0^{\infty} x \cdot \frac{15}{2}(e^{-3x}-e^{-5x})dx = \frac{8}{15}, \\
EY &= \int_0^{\infty} y \cdot (5e^{-5y})dy = \frac{1}{5}, \\
EXY &= 15\int_0^{\infty}\int_0^{x} xy \cdot e^{-3x}e^{-2y}dydx = \frac{11}{75}, \\
EX^2 &= \int_0^{\infty} x^2 \cdot \frac{15}{2}(e^{-3x}-2e^{-5x})dx = \frac{98}{225}, \\
EY^2 &= \int_0^{\infty} y^2 \cdot (5e^{-5y})dy = \frac{2}{25}.
\end{aligned}$$

Then $Cov(X,Y) = EXY - EXEY = \frac{11}{75} - \frac{8}{15}\cdot\frac{1}{5} = \frac{1}{25}$, and

$$\rho = \frac{Cov(X,Y)}{\sigma_x\sigma_y} = \frac{EXY - EXEY}{\sqrt{EX^2-(EX)^2}\cdot\sqrt{EY^2-(EY)^2}} = \frac{3}{\sqrt{34}} \approx 0.51$$

**Problem 8.3**

Assume the random variables $X$ and $Y$ representing marks obtained in Midterm I and Midterm II in a course are jointly normally distributed and that

$$\mu_X = 65, \qquad \sigma_X = 18, \qquad \mu_Y = 60, \qquad \sigma_Y = 20,$$

$$\rho \,\underline{\Delta}\, (\sigma_X\sigma_Y)^{-1}[E(x-\mu_X)(y-\mu_y)] \equiv (\sigma_X\sigma_Y)^{-1}\sigma_{X,Y} = 0.75.$$

(i) Verify that $\rho\,\frac{\sigma_Y}{\sigma_X} = \frac{\sigma_{X,Y}}{\sigma_X^2}$. (Use the expression on the left or the right of this equality in the work below depending upon which you find most convenient.)

(ii) *Assume* that the conditional mean of $Y$ given that $X = x$ is given by the formula

$$\mu_{Y|X} \triangleq E(Y|X = x) = (\mu_Y + \rho\tfrac{\sigma_Y}{\sigma_X}(x - \mu_X)).$$

Find the value of $\quad E_X[E(Y|X = x)] \equiv \int_{-\infty}^{\infty} E(Y|X = x)p(x)dx.$

(iii) (a) Assume a student's mark in Midterm I is $X = 75$. Use the formula in (ii) to find the conditional mean of the student's mark $E(Y|X = 75)$ in Midterm II.

(b) Assume a student's mark in Midterm I is $X = 81$; find also in this case the conditional mean of the student's mark in Midterm II.

**Solution 8.3**

(i)

$$\rho\frac{\sigma_Y}{\sigma_X} = (\sigma_X\sigma_Y)^{-1}\sigma_{X,Y}\cdot\frac{\sigma_Y}{\sigma_X} = \frac{\sigma_{X,Y}}{\sigma_X^2}$$

(ii)

$$\begin{aligned}
E_X[E(Y|X = x)] &= \int_{-\infty}^{\infty} E(Y|X = x)p(x)dx \\
&= \int_{-\infty}^{\infty} \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X))p(x)dx \\
&= \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}\cdot E_X(x - \mu_X) \\
&= \mu_Y + 0 = \mu_Y
\end{aligned}$$

(iii)

(a) From (ii),

$$E(Y|X = 75) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(75 - \mu_X) = 60 + 0.75\cdot\frac{20}{18}(75 - 65) \approx 68.3$$

(b) Similarly,

$$E(Y|X = 81) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(81 - \mu_X) = 60 + 0.75\cdot\frac{20}{18}(81 - 65) \approx 73.3$$

**Problem 8.4**

A Poission RV with parameter $\lambda > 0$ takes values in the postive integers $N$; its probability mass function is given by:

$$P(X = n) = e^{-\lambda}\frac{\lambda^n}{n!}, \qquad n \in N.$$

The number of times that a person contracts a cold in a given year is described by a Poisson random random variable with parameter $\lambda = 4$. Suppose that a new wonder drug (based on large quantities of vitamin C) has just been marketed that reduces the Poisson parameter to $\lambda = 3$ for 60% of the population. For the other 40% of the population the drug has no appreciable effects on colds.

A certain individual tries the drug for a year and has 2 colds in that time; use Bayes theorem and the Poisson distribution to determine how likely it is that the drug is beneficial for him or her (i.e. find the probability that the individual falls in the first group).

**Solution 8.4**

Let $P(A) = P(\text{drug is beneficial}) = 0.6$ and let the random variable $X$ be the number of cold the person contracts in a year. Then $P(\text{individual falls in the first group}) = P(A|X = 2)$. Using Bayes theorem and have that

$$P(A|X = 2) = \frac{P(X = 2|A)P(A)}{P(X = 2)},$$

where

$$\begin{aligned} P(X = 2) &= P(X = 2|A)P(A) + P(X = 2|A^C)P(A^C) \\ &= \frac{e^{-3}3^2}{2!} \times 0.6 + \frac{e^{-4}4^2}{2!} \times 0.4 \end{aligned}$$

and

$$P(X = 2|A) = \frac{e^{-3}3^2}{2!}.$$

Thus, we have $P(A|X = 2) \approx 0.6964$.