ECSE 304-305B Assignment 7 FALL 2008

7.1 Solution:

(i) Since X_i are independent random variables and uniformly distributed on the interval $[-k, k]$, the characteristic functions are given by

$$\Phi_{X_i}(w) = \int_{-k}^{k} e^{jwx} \frac{1}{2k} dx = \frac{e^{jwk} - e^{-jwk}}{2kjw},$$

which are even functions since $\Phi_{X_i}(w) = \Phi_{X_i}(-w)$.

Given $Y_n = \sum_{k=1}^{n} X_k$,

$$\begin{aligned} \Phi_{Y_n}(w) &= E(\exp^{jwY_n}) = E(\exp^{jw(X_1+X_2+\cdots+X_n)}) \\ &= E(\exp^{jwX_1})E(\exp^{jwX_2})\cdots E(\exp^{jwX_n}) \\ &= \Phi_{X_1}(w)\Phi_{X_2}(w)\cdots\Phi_{X_n}(w). \end{aligned}$$

Given $Z_n = \sum_{k=1}^{n}(-1)^k X_k$,

$$\begin{aligned} \Phi_{Z_n}(w) &= E(\exp^{jwZ_n}) = E(\exp^{jw(-X_1+X_2+\cdots+(-1)^n X_n)}) \\ &= E(\exp^{-jwX_1})E(\exp^{jwX_2})\cdots E(\exp^{(-1)^n jwX_n}) \\ &= \Phi_{X_1}(-w)\Phi_{X_2}(w)\cdots\Phi_{X_n}((-1)^n w) \end{aligned}$$

Since each Φ_i is an even function of w, we have

$$\Phi_{Z_n}(w) = \Phi_{X_1}(w)\Phi_{X_2}(w)\cdots\Phi_{X_n}(w),$$

that is to say $\Phi_{Z_n}(w) = \Phi_{Y_n}(w)$.

And because distribution functions are in one to one relation with characteristic functions, we conclude that Y_n and Z_n have identical distributions.

(ii) As $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$ we have

$$\Phi_{\infty}(w) = e^{\mu jw - \frac{w^2\sigma^2}{2}}.$$

(iii) It is a Gaussian distribution function whereas

$$f_{\infty}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

7.2 Solution:

(i)

$$\Phi_{T_\lambda}(w) = \int_{-\infty}^{\infty} e^{jwt} \cdot f_t(t)\, dt = \int_{0}^{\infty} e^{jwt} \cdot \lambda \cdot e^{-\lambda t} dt = \frac{\lambda}{\lambda - jw}.$$

(ii) Similar to part (i), we have

$$\Phi_{T_\mu}(w) = \frac{\mu}{\mu - jw}.$$

Since T_λ and T_μ are independent random variables, the characteristic function of the total waiting time $T_\lambda + T_\mu$ is

$$\Phi_{T_\lambda + T_\mu}(w) = \Phi_{T_\lambda}(w) \cdot \Phi_{T_\mu}(w) = \frac{\lambda\mu}{(\lambda - jw)(\mu - jw)}.$$

(iii)

$$\Phi_{2T_\lambda}(w) := E\left[e^{jw2T_\lambda}\right] = E\left[e^{j(2w)T_\lambda}\right] = \Phi_{T_\lambda}(2w) = \frac{\lambda}{\lambda - j(2w)}.$$

(iv)

$$\begin{aligned} \Phi_{T_\lambda + T_\mu}(w)\Big|_{\mu=\lambda} &= \frac{\lambda\mu}{(\lambda - jw)(\mu - jw)}\Bigg|_{\mu=\lambda} = \frac{\lambda^2}{(\lambda - jw)^2} \\ &= \frac{\lambda^2}{\lambda^2 - 2j\lambda\mu - w^2} \neq \Phi_{2T_\lambda}(w). \end{aligned}$$

Hence their density functions could not be the same, since

$$f_x(x) = (1/2\pi) \int_{-\infty}^{\infty} \Phi_x(w) \cdot \exp\left(-jwx\right) dw.$$

(v)

$$\begin{aligned} EX^2 &= \frac{1}{j^2} \frac{d^2}{d\, w^2} \Phi_x(w)\Bigg|_{w=0} \\ &= \frac{2}{\mu^2} + \frac{2}{\lambda^2} + \frac{2}{\mu\lambda}. \end{aligned}$$

7.3 Solution:

(i) By CLT,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i}{\sqrt{4}} \approx N(0,1)$$

Therefore,

$$\begin{aligned} P(-\alpha \le \frac{1}{n}\sum_{i=1}^{n}X_i \le \alpha) &= P(-\frac{\alpha\sqrt{n}}{2} \le \frac{1}{\sqrt{4n}}\sum_{i=1}^{n}X_i \le \frac{\alpha\sqrt{n}}{2}) \\ &\approx \Phi(\frac{\alpha\sqrt{n}}{2}) - \Phi(-\frac{\alpha\sqrt{n}}{2}) \\ &= 2\Phi(\frac{\alpha\sqrt{n}}{2}) - 1 \end{aligned}$$

(ii) When α=1:

(a) $2\Phi(\frac{\sqrt{n}}{2}) - 1 \ge 0.95 \Rightarrow \Phi(\frac{\sqrt{n}}{2}) \ge 0.975 \Rightarrow \frac{\sqrt{n}}{2} \ge 1.96 \Rightarrow n \ge 15.37$

Since n is an integer, it has to be equal to or bigger than 16.

(b) $2\Phi(\frac{\sqrt{n}}{2}) - 1 \ge 0.9786 \Rightarrow \Phi(\frac{\sqrt{n}}{2}) \ge 0.9893 \Rightarrow \frac{\sqrt{n}}{2} \ge 2.3 \Rightarrow n \ge 21.16$

Since n is an integer, it has to be equal to or bigger than 22.

7.4 Solution:

(a)

$$\begin{aligned} \Phi_Z(\omega) &= \int_{-\infty}^{\infty} f_Z(z)e^{j\omega z} = \int_0^{\infty} 5\mu e^{(j\omega - 5\mu z)}dz \\ &= \frac{5\mu}{j\omega - 5\mu} \cdot e^{z(j\omega-5\mu)}|_0^{\infty} = \frac{5\mu}{j\omega - 5\mu} \end{aligned}$$

(b)

$$\begin{aligned} \Phi_X(\omega) &= \int_{-\infty}^{\infty} f_X \cdot e^{j\omega x}dx = \int_{-\infty}^{0} \frac{\lambda}{2}e^{x(j\omega+\lambda)}dx + \int_0^{\infty} \frac{\lambda}{2}e^{x(j\omega-\lambda)}dx \\ &= \frac{\lambda}{2(j\omega+\delta)} + \frac{\lambda}{2(j\omega-\lambda)} = \frac{\lambda^2}{\omega^2+\lambda^2} \end{aligned}$$

(c) From the Fourier transform we can get $\Phi_{-Z}(\omega) = \Phi_Z(-\omega)$

As Z_1 and Z_2 are independent $\Phi_W(\omega) = \Phi_{Z_1}(\omega) \cdot \Phi_{Z_2}(\omega) = \frac{\lambda^2}{\lambda^2+\omega^2}$

(d)From the one-to-one relationship, and the characteristic function got above, the probability density of W is $f_W(w) = \frac{\lambda}{2}e^{-\lambda|w|}$