ECSE 304-305 Assignment 7 Fall 2008

Return by 12.00 pm, 6th November, 2008

7.1

(i) Consider a sequence of independent random variables $\{X_k; 1 \leq k\}$ such that X_k is uniformly distributed on the interval $[-k, k]$. Define:

$$Y_n = \sum_{k=1}^{n} X_k, \qquad Z_n = \sum_{k=1}^{n} (-1)^k X_k$$

By use of characteristic functions show whether or not Y_n and Z_n have identical distributions for all $\{n; 1 \leq n\}$.

(ii) Give the characteristic function $\Phi_\infty(\omega)$, $\omega \in R$, which is the limit of the sequence of characteristic functions

$$\Phi_n(\omega) = (1 + \frac{1}{n}[\mu j\omega - \frac{\omega^2\sigma^2}{2}])^n, \quad \omega \in R, \qquad n \longrightarrow \infty.$$

(iii) Find the probability density $f_\infty(\cdot)$ corresponding to $\Phi_\infty(\omega)$, $\omega \in R$.

7.2

(i) Give the characteristic function of the exponential waiting time distribution on $[0, \infty)$ with parameter $\lambda > 0$.

A traveller at Trudeau International Airport must wait in two queues in series: first, the traveller must wait at the Check-in queue for his or her airline; this has an exponentially distributed waiting time T_λ, with parameter $\lambda > 0$; second, the traveller must wait in a queue in the Security Zone with an exponentially distributed waiting time T_μ, with parameter $\mu > 0$.

It is assumed that T_λ and T_μ are independent random variables.

(ii) What is the characteristic function of the total waiting time $T_\lambda + T_\mu$?

(iii) Find the characteristic function of $2T_\lambda$.

(iv) By use of characteristic functions, or otherwise, show whether the density of $T_\lambda + T_\mu$ with $\mu = \lambda$ is the same as that of $2T_\lambda$.

(v) Find the second moment of $T_\lambda + T_\mu$.

7.3

Assume each of the independent identically distributed scalar random variables $X_i, 1 \leq i < \infty$, has mean 0 and variance $\sigma^2 = 4$. For $\alpha > 0$, consider the probability of the event :

$$A \;=\; \{-\alpha \leq \tfrac{1}{n}\textstyle\sum_{i=1}^{n} X_i \leq \alpha\}$$

(i) Use the Central Limit Theorem, together with the notation $\Phi(x), x \in R$, for the distribution function of a normally distributed $N(0,1)$ random variable, to give a formula for an approximation to the probability that the average $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ lies in the interval $[-\alpha, \alpha]$.
(ii) Let $\alpha = 1$. Use the CLT based formula to find the smallest value of n for which the probability of A is at least: (a) 0.95 and (b) 0.9786. (You may use the fact that for the Gaussian distribution $\Phi(-x) = 1 - \Phi(x), x \in R$, and may use any standard Gaussian distribution table; for instance in the course text this is given on page SG 632.)

7.4

(a) The exponential random variable Z has the density $f_Z(\cdot) = \{5\mu e^{-5\mu z}, z \in R_+, \mu > 0\}$. Find the characteristic function $\Phi_Z(\omega), \omega \in R$.

(b) The random variable X has a Laplacian density $\frac{\lambda}{2}e^{-\lambda|x|}, x \in R, \lambda > 0$. Find the characteristic function $\Phi_X(\omega), \omega \in R$, of X.

(c) Let the random variable W be defined by $W = Z_1 - Z_2$, where Z_1 and Z_2 are independent identically distributed exponential random variables with parameter λ. Find $\Phi_W(\omega), \omega \in R$.

(d) Using the one-to-one relation of characteristic functions and densities and part (c), find the density of W.