ECSE 304-305B Assignment 6 Solutions Fall 2008

Question 6.1

A scalar random variable X is sent through a multiplicative coding channel which generates the output $Z = XY$: The independent positive scalar random variables X and Y have the densities $\{f_X(x) = x^{-2}, 1 \le x < \infty\}$ and $\{f_Y(y) = 2y^{-3}, 1 \le y < \infty\}$ respectively.

(a) Find the densities of the random variables: $X' = logX$, $1 \le X < \infty$, and $Y' = logY$, $1 \le Y < \infty$; on $R_+ = [0, 1)$.

(b) Find the characteristic functions of the densities of X', Y' and $Z' = \log Z$.

(c) Give the probability density of the coded version Z of the signal X.

Q 6.1 Solution:

(a) Given $X' = \log X$ and $Y' = \log Y$, find $f_{X'}(x')$ and $f_{Y'}(y')$ respectively.

By definition, the densities for X' and Y' are defined as follows:

$$f_{X'}(x') = f_X(x(x')) \mid \frac{\partial x(x')}{\partial x'} \mid$$
$$f_{Y'}(y') = f_Y(y(y')) \mid \frac{\partial y(y')}{\partial y'} \mid$$

It follows that

$$x'(x) = \log x \;\Rightarrow x(x') = e^{x'} \Rightarrow \frac{\partial x(x')}{\partial x'} = e^{x'}$$
$$y'(y) = \log y \;\Rightarrow y(y') = e^{y'} \Rightarrow \frac{\partial y(y')}{\partial y'} = e^{y'}$$

and

$$
\begin{aligned}
f_X(x(x')) &= (e^{x'})^{-2} &= e^{-2x'}, \quad 0 \le x' < \infty \\
f_Y(y(y')) &= 2(e^{y'})^{-3} &= 2e^{-3y'}, \quad 0 \le y' < \infty
\end{aligned}
$$

Then the final densities will be as follows:

$$
\begin{aligned}
f_{X'}(x') &= e^{-2x'}e^{x'} &= e^{-x'}, \quad 0 \le x' < \infty \\
f_{Y'}(y') &= 2e^{-3y'}e^{y'} &= 2e^{-2y'}, \quad 0 \le y' < \infty
\end{aligned}
$$

(b) Find the characteristic functions of the densities of X', Y' and $Z' = \log Z$.

Following the definition of the characteristic function

$$
\begin{aligned}
\Phi_{X'}(\omega) &= \int_{-\infty}^{\infty} e^{j\omega x'} f_{X'}(x')dx' \\
\Phi_{Y'}(\omega) &= \int_{-\infty}^{\infty} e^{j\omega y'} f_{Y'}(y')dy'
\end{aligned}
$$

Since x' and y' are defined in certain ranges, and substituting for $f_{X'}(x')$ and $f_{Y'}(y')$ respectively, the characteristic functions become as follows:

$$
\begin{aligned}
\Phi_{X'}(\omega) &= \int_{0}^{\infty} e^{j\omega x'} e^{-x'} dx' &= \frac{1}{(1-j\omega)} \\
\Phi_{Y'}(\omega) &= \int_{0}^{\infty} e^{j\omega y'} 2e^{-2y'} dy' &= \frac{2}{(2-j\omega)}
\end{aligned}
$$

For $Z' = \log Z$, its expression can be simplified as follows:

$$
\begin{aligned}
Z' &= \log Z \\
&= \log XY = \log X + \log Y = X' + Y'
\end{aligned}
$$

which is simply the summation of two independent random variables. The characteristic function for Z', by definition, will be as follows:

$$\begin{aligned}\Phi_{Z'}(\omega) &= \Phi_{(X'+Y')}(\omega) \\ &= \Phi_{X'}(\omega)\Phi_{Y'}(\omega) \\ &= \frac{1}{(1-j\omega)}\frac{2}{(2-j\omega)} \\ &= \frac{2}{(1-j\omega)(2-j\omega)}, \qquad \omega \in R\end{aligned}$$

(c) Give the probability density of the coded version Z of the signal X.

Using the property of characteristic functions that they have unique inverses, we can restore the density $f_{Z'}(z')$ from $\Phi_{Z'}(\omega)$ then use the transformation $Z = e^{Z'}$ to restore the original density $f_Z(z)$. First, let us expand the above solution for $\Phi_{Z'}(\omega)$ and using partial fraction to see it clearly.

$$\begin{aligned}\Phi_{Z'}(\omega) &= \frac{2}{(1-j\omega)(2-j\omega)} \\ &= \frac{a}{(1-j\omega)} + \frac{b}{(2-j\omega)} \\ &= \frac{2}{(1-j\omega)} + \frac{-2}{(2-j\omega)} \\ \Phi_{(X'+Y')}(\omega) &= 2\Phi_{X'}(\omega) - \Phi_{Y'}(\omega)\end{aligned}$$

By the inverse Fourier transform, we know that $\Phi_{(X'+Y')}(\omega)$ must be the characteristic function of the density $f_{X'+Y'}(x'+y')$ which is actually the density $f_{Z'}(z')$. Therefore,

$$f_{Z'}(z') = 2e^{-z'} - 2e^{-2z'}, \qquad z' \geq 0$$

Using the transformation rule

$$f_Z(z) = f_{Z'}(z'(z))|\frac{\partial z'(z)}{\partial z}|,$$

$$z'(z) = \log z \quad \Rightarrow \quad \frac{\partial z'(z)}{\partial z} = 1/z$$

$$f_{Z'}(z'(z)) = 2e^{-\log z} - 2e^{-2\log z}$$

This makes the final density

$$f_Z(z) = (\frac{2}{z} - \frac{2}{z^2})\frac{1}{z}, \qquad z \geq 1$$

You can verify the answers by integrating $f_Z(z)$ from 1 to ∞ and $f_{Z'}(z')$ from 0 to ∞.

Question 6.2

Find (a) the mean value μ, and (b) the variance σ^2 of an RV X with the *Laplace density*

$$f_X(x) = \frac{1}{2b}e^{-2|x-m|/2b},$$

where b and m are real constants, $b > 0$ and $-\infty < m < \infty$.

Find the corresponding characteristic function $\Phi_X(\omega)$ and verify the values found above for $\mu,\ \sigma^2$ by use of the Moment Theorem.

Q 6.2 Solution:

We can find the mean and variance of X from a table or by integrating

$$
\begin{aligned}
EX &= \int_{-\infty}^{\infty} (x - m + m) f_X(x) dx \\
&= \int_{-\infty}^{\infty} (x - m) \frac{1}{2b} e^{-2|x-m|/2b} dx + \int_{-\infty}^{\infty} m f_X(x) dx \\
&= \int_{-\infty}^{\infty} y \frac{1}{2b} e^{-2|y|/2b} dy + m \\
&= 0 + m \quad \text{(by symmetry of the integrand).}
\end{aligned}
$$

Integration by parts can be used to obtain EX^2. In the end, we find:

$$\mu = m,$$

$$\sigma^2 = 2b^2.$$

The characteristic function is given by:

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x) dx = \int_{-\infty}^{\infty} e^{j\omega x} \tfrac{1}{2b} e^{-2|x-m|/2b} dx.$$

Using the hint, we set the parameter m=0, making X a zero-mean R.V. The characteristic function in this case is given by:

$$
\begin{aligned}
\Phi_X(\omega) &= \int_{-\infty}^{\infty} e^{j\omega x}\frac{1}{2b}e^{-2|x|/2b}dx \\
&= \frac{1}{2b}\int_{-\infty}^{0} e^{x(\frac{1}{b}+j\omega)}dx + \frac{1}{2b}\int_{0}^{\infty} e^{x(-\frac{1}{b}+j\omega)}dx \\
&= \frac{1}{2b}\left(\frac{1}{\frac{1}{b}+j\omega}e^{x(\frac{1}{b}+j\omega)}|_{-\infty}^{0} - \frac{1}{-\frac{1}{b}+j\omega}e^{x(-\frac{1}{b}+j\omega)}|_{0}^{\infty}\right) \\
&= \frac{1}{2b}\left(\frac{1}{\frac{1}{b}+j\omega} + \frac{1}{\frac{1}{b}-j\omega}\right) \\
&= \frac{1}{1+b^2\omega^2}
\end{aligned}
$$

Now for a R.V. X of mean m, and using the formula on page 8 of the Lecture to account for the shift, the characteristic function becomes:

$$
\Phi_X(\omega) = \frac{e^{jm\omega}}{1+b^2\omega^2}.
$$

By the moment theorem,

$$
\begin{aligned}
EX &= (1/j)\frac{d}{d\omega}\frac{e^{jm\omega}}{1+b^2\omega^2}\Big|_{\omega=0} \\
&= (1/j)\left(\frac{-2b^2\omega}{(1+b^2\omega^2)^2}e^{jm\omega} + \frac{jme^{jm\omega}}{1+b^2\omega^2}\right)\Big|_{\omega=0} \\
&= m.
\end{aligned}
$$

By taking one more derivative, we find

$$
\begin{aligned}
EX^2 &= -\frac{d}{d\omega}\Big(\frac{-2b^2\omega}{(1+b^2\omega^2)^2}e^{jm\omega} + \frac{jme^{jm\omega}}{1+b^2\omega^2}\Big)\Big|_{\omega=0} \\
&= -\Big(\frac{-(2b^2)(1+b^2\omega^2)^2 + 2b^2\omega(1+b^2\omega^2)(2b^2\omega)}{(1+b^2\omega^2)^4}e^{jm\omega} + \frac{-2b^2\omega}{(1+b^2\omega^2)^2}(jm)e^{jm\omega} \\
&\quad + \frac{-2b^2\omega}{(1+b^2\omega^2)^2}(jm)e^{jm\omega} + \frac{1}{(1+b^2\omega^2)^2}(jm)^2e^{jm\omega}\Big)\Big|_{\omega=0} \\
&= 2b^2 + m^2.
\end{aligned}
$$

Hence, the variance is $2b^2$.

Question 6.3

(a) The exponential random variable Z has the density $f_Z(\cdot) = \{5\mu e^{-5\mu z}, z \ \in R_+, \mu > 0\}$. Find the characteristic function $\Phi_Z(\omega), \omega \in R$.

(b) Let the random variable W be defined by $W = 3(Z_1) - 3(Z_2)$, where Z_1 and Z_2 are independent identically distributed exponential random variables with parameter λ. Find $\Phi_W(\omega), \omega \in R$.

(c) Using the one-to-one relation of characteristic functions and densities and part (b), find the density of W. (Hint: check Question 2.)

Q 6.3 Solution:

(a) We can obtain the characteristic function from a table or by integration. Let's integrate.

$$\Phi_Z(w) = \int_0^\infty e^{jtx} 5\mu e^{-5\mu x} dx = \frac{5\mu}{5\mu - jw}.$$

(b) First, observe that the characteristic function of $-3Z_2$ is

$$\Phi_{-3Z_2}(w) = \Phi_{Z_2}(-3w),$$

by the scaling property seen earlier. By independence, we have

$$\begin{aligned}\Phi_W(w) &= \Phi_{3Z_1}(w)\Phi_{-3Z_2}(w)\\ &= \Phi_{Z_1}(3w)\Phi_{Z_2}(-3w)\\ &= \frac{\lambda}{\lambda - j3w}\frac{\lambda}{\lambda + j3w}\\ &= \frac{\lambda^2}{\lambda^2 + 9w^2} = \Phi_X(w).\end{aligned}$$

(c) Since $\Phi_W(t) = \Phi_X(t)$, by the inversion formula, X and W have the same distribution if we let $\lambda = \frac{3}{b}$ and $m = 0$. Hence, for all x

$$f_W(x) = \frac{\lambda e^{-\frac{\lambda|x|}{3}}}{6}$$

.

Question 6.4

X is a scalar random variable distributed $N(\mu, \sigma^2) = N(1, 4),\ \sigma > 0$. Let $Z = g(X) = \frac{X^2}{2}$. Find the density $F_Z(z)$ of Z for $z > 0$.

Q 6.4 Solution:

X is a random variable with Gaussian distribution $N(1, 4)$, so

$$f_X(x) = \frac{1}{2\sqrt{2\pi}}e^{-\frac{(x-1)^2}{8}}.$$

Since $g(x) = \frac{x^2}{2}$ then $g^{-1}(x) = \pm\sqrt{2x}$. we know

$$f_Z(z) = \sum_{g^{-1}(z)\in A_i} f_X(g^{-1}(z)).|\frac{dg^{-1}(z)}{dz}|.$$

Therefore

$$f_Z(z) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(\sqrt{2z}-1)^2}{8}} \times \frac{1}{\sqrt{2z}} + \frac{1}{2\sqrt{2\pi}} e^{-\frac{(-\sqrt{2z}-1)^2}{8}} \times \frac{-1}{\sqrt{2z}}$$

which equals to

$$f_Z(z) = \frac{1}{4\sqrt{\pi z}} \left[e^{-\frac{(\sqrt{2z}-1)^2}{8}} - e^{-\frac{(-\sqrt{2z}-1)^2}{8}} \right],$$

for $z \geq 0$ and 0 otherwise.

Question 6.5

Many people believe that the daily change of price of a company's stock on the stock market is a random variable with mean 0 and variance σ^2. That is, if Y_n represents the price of the stock on the nth day, then

$$Y_n = Y_{n-1} + X_n \qquad n \geq 1$$

where $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean 0 and variance σ^2. Suppose that the stock's price today is 100. If $\sigma^2 = 1$, what can you say about the probability that the stock's price will exceed 105 after 10 days?

Q 6.5 Solution:

We have $Y_{10} = Y_0 + \sum_{i=1}^{10} X_i$, $E(Y_{10}) = \mu = 100$ and $E((Y_{10} - \mu)^2) = \sigma^2 = 10$. We wish to apply Chebyshev's inequality

$$P(|Y_{10} - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Noting the symmetry of Y_{10}, we can half Chebyshev's inequality, drop the absolute value and hence obtain a bound for $P(Y_{10} > 105)$ (with $k = \frac{5}{\sqrt{10}}$)

$$P(Y_{10} - 100 \geq 5) \leq \frac{1}{2k^2} = \frac{1}{5}$$