ECSE 304-305B Assignment 4 Solutions Fall 2008

Question 4.1 [15]

Let X be a random point be selected from the interval $0, 3)$ (i.e. by the EPP, or, equivalently, such that X has a linearly increasing distribution function). What is the probability that (i)$X^2 - 5X + 6 > 0$, and (ii)$X^2 - 5X + 6 \leq 1$

Solution:

(i)$X^2 - 5X + 6 > 0$ We have that

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{3} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

and since

$$P(X^2 - 5X + 6 > 0) = 1 - P(2 \leq X \leq 3)$$

we can compute $P(X^2 - 5X + 6 > 0) = 1 - (F_X(3) - F_X(2)) = \frac{2}{3}$

(ii)$X^2 - 5X + 6 \leq 1$ We can compute

$X^2 - 5X + 6 = 1 \Rightarrow x = \frac{5 \pm \sqrt{5}}{2}$

So $P(X^2 - 5X + 6 \leq 1) = F_X(3) - F_X(\frac{5-\sqrt{5}}{2}) = \frac{1+\sqrt{5}}{6}$

Question 4.2 [15]

A student takes a certain type of test four times and the student's final score will be the maximum of the test score. Thus

$$\mathbf{X} = max\{\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}, \mathbf{X_4}\}, \quad (1)$$

where $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}, \mathbf{X_4}$ are the four test scores and $\mathbf{X}$ is the final score. Assume the score takes values $x, 0 \leq x \leq 100$, and the results of the tests form an independent set of random variables, each with distribution function $F_{X_i} = F(x) = P(\mathbf{X_i} \leq x), \quad 1 \leq i \leq 4$. Let the distribution function of $\mathbf{X}$ be written as

$$F_{\mathbf{X}}(x) = \{P(\mathbf{X} \leq x), \quad 0 \leq x \leq 100\}. \tag{2}$$

(a) Find which of the following gives $F_{\mathbf{X}}(\cdot)$ and give the derivation:

(i) $F^{1/4}(x^4)$, (ii) $F^4(x)$, (iii) $F(x^4)$

(b) If $F(x) = \{0, x < 0; \ \frac{x}{100}, 0 \leq x \leq 100; \ 1, 100 \leq x\}$, and given that any score $\mathbf{X}_i$ is uniformly distributed over $[0, 100]$, find $F_{\mathbf{X}}(80) = Pr\{\mathbf{X} \leq 80\}$.

Solution:

(a) We have that $\mathbf{X} = max\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\, \mathbf{X}_4\}$. Noting that the event $\{x \leq \mathbf{X}\}$ is equivalent to the event $\{\mathbf{X}_1 \leq x\} \cap \{\mathbf{X}_2 \leq x\} \cap \{\mathbf{X}_3 \leq x\} \cap \{\mathbf{X}_4 \leq x\}$ we have that

$$\begin{aligned}
F_{\mathbf{x}}(x) &= \{P(\mathbf{X} \leq x), 0 \leq x \leq 100\} \\
&= P(\max\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\, \mathbf{X}_4\} \leq x) \\
&= P(\{\mathbf{X}_1 \leq x\} \cap \{\mathbf{X}_2 \leq x\} \cap \{\mathbf{X}_3 \leq x\} \cap \{\mathbf{X}_4 \leq x\}) \\
&= \prod_{i=1}^{4} P(\mathbf{X}_i \leq x), \quad \text{since } \mathbf{X}_i\text{'s are independent} \\
&= F^4(x).
\end{aligned}$$

(b) $F_{\mathbf{X}}(80) = P(\mathbf{X} \leq 80) = F^4(80) = \left(\frac{80}{100}\right)^4 = \frac{256}{625} = 0.4096$

Question 4.3 [15]

For a constant parameter $a > 0$, a Rayleigh random variable X has PDF

$$f_X(x) = \begin{cases} a^2 x e^{-a^2x^2/2} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the CDF of X?

Solution:

We know

$$F_X(x) = \int_{-\infty}^{x} f_X(y)dy.$$

For $x \leq 0$ we have $F_X(x) = 0$ since $f_X(x) = 0$. For $x > 0$

$$F_X(x) = \int_0^x [a^2 y e^{-a^2y^2/2}]dy = -e^{-a^2y^2/2}|_0^x = 1 - e^{-a^2x^2/2}.$$

Therefore

$$F_X(x) = \begin{cases} 1 - e^{-a^2x^2/2} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Question 4.4 [20]

The (cumulative) distribution function of the random variable U is

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \leq u < -3, \\ 1/4 & -3 \leq u < 3, \\ 1/4 + 3(u-3)/8 & 3 \leq u < 5, \\ 1 & u \geq 5. \end{cases}$$

(a) What is $E[U]$?

(b) What is $Var[U]$?

(c) What is $E[2^U]$?

Solution:

(a) By definition $E[U] = \int_{-\infty}^{\infty} uf(u)du$. We find that the density function $f(u)$ has the following values:

$$f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3, \\ 0 & -3 \le u < 3, \\ 3/8 & 3 \le u < 5, \\ 0 & u \ge 5. \end{cases}$$

$$\begin{aligned} E[U] &= \frac{1}{8}\int_{-5}^{-3} udu + \frac{3}{8}\int_{3}^{5} udu \\ &= 2 \end{aligned}$$

(b) We want to compute $Var[U] = E[U^2] - (E[U])^2$. We know $E[U]$ so all we need to find is $E[U^2]$. Using $E[g(U)] = \int_{-\infty}^{\infty} g(u)f_U(u)du$ we get

$$\begin{aligned} E[U^2] &= \int_{-\infty}^{\infty} u^2 f_U(u)du \\ &= \frac{49}{3} \end{aligned}$$

Therefore,

$$Var[U] = \frac{49}{3} - 2^2 = \frac{37}{3}$$

(c)

$$\begin{aligned} E[2^U] &= \int_{-\infty}^{\infty} 2^u f_U(u) du \\ &= \left[\frac{2^u}{8ln2}\right]_{-5}^{-3} + \left[\frac{3(2^u)}{8ln2}\right]_3^5 \\ &= \frac{2307}{256ln2} \simeq 18 \end{aligned}$$

Question 4.5 [20]

X is a continuous uniform (-4,4) random variable.

(a) What is the PDF $f_X(x)$?

(b) What is the CDF $F_X(x)$?

(c) What is $E[X]$?

(d) What is $E[X^5]$?

(e) What is $E[e^X]$?

Solution:

X is a uniform random variable with $a = -4$ and $b = 4$. (a)

$$f_X(x) = \begin{cases} \frac{1}{8} & -4 \leq x < 4, \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$F_X(x) = \begin{cases} 0 & x < -4, \\ \frac{x}{8} + \frac{1}{2} & -4 \le x < 4, \\ 1 & x > 4. \end{cases}$$

(c)

$$\begin{aligned} E[X] &= \frac{a+b}{2} \\ &= 0 \end{aligned}$$

(d)

$$\begin{aligned} E[X^5] &= \int_{-\infty}^{\infty} x^5 f(x)dx \\ &= \int_{-4}^{4} \frac{x^5}{8} dx \\ &= 0 \end{aligned}$$

remark:

$$E[X^n] = \begin{cases} 0 & n : \text{odd}, \\ \dfrac{4^n}{n+1} & n : \text{even}. \end{cases}$$

(e)

$$\begin{aligned} E[e^x] &= \int_{-\infty}^{\infty} e^x f(x)dx \\ &= \int_{-4}^{4} \frac{e^x}{8} dx \\ &= \frac{e^4 - e^{-4}}{8} = \frac{\sinh 4}{4}. \end{aligned}$$

Question 4.6 [15]

The time until a small meteorite first lands anywhere in the Sahara desert is modeled as an exponential random variable with a mean (= the recipricol of the rate λ) of 10 days. The time is currently midnight. What is the probability (to two decimal places) that a meteorite first lands some time between 6 a.m. and 6 p.m. of the first day?

Solution:

Let the random variable X denote the meteorite's landing time expressed in hours. It is given that X is modelled as a random variable with a probability density defined as follows:

$$X \sim f(X) = \begin{cases} \lambda \exp\{-\lambda\ X\} & \text{if } X \geq 0 \\ 0 & \text{if } X < 0 \end{cases}$$

It is given that the mean of the distribution, or $E[X] = 10$ days, i.e. 240 hours. Since for an exponentially distributed random variable

$$E[X] = \frac{1}{\lambda} \quad \Longrightarrow \quad \lambda = 0.0042$$

Given that it is currently midnight (or 00:00 time), it is requested to know the probability that the meteorite will land at any time between 06:00 *am* and 06:00 *pm* (or 18:00). This can be expressed in terms of probability as follows: $\Pr(x_1 = 6 < X \leq x_2 = 18)$ where,

$$\Pr(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f(X)\ dX \tag{3}$$

For an exponential random variable:

$$
\begin{aligned}
\Pr(x_1 < X \le x_2) &= \lambda \int_{x_1}^{x_2} \exp\{-\lambda\ X\}\ \ dX \\
&= \lambda \left[-\frac{1}{\lambda} \exp\{-\lambda\ X\} \right]_{x_1}^{x_2} \\
&= -\exp\{-\lambda\ x_2\} + \exp\{-\lambda\ x_1\} \\
\Pr(6 < X \le 18) &= 0.0479
\end{aligned}
$$