ECSE 304-305B Assignment 10 Solutions Fall 2008

Question 10.1 (Random Phase Shifted Signal)

The scalar sinusoidal signal $x(t) = sin(\theta t), t \in \mathbb{R}$, passes through a channel C where it is distorted by a constant (in time) random additive phase change between the input and the output of C. This produces the random signal

$$y(t) = sin(\theta t + w), \qquad t \in \mathbb{R},$$

where the random disturbance w is independent of time and is uniformly distributed on $[0,\ 2\pi]$, i.e. $w \sim U[0, 2\pi]$.

By explicitly calculating (a) the mean of the output process $Ey(t), t \in \mathbb{R}$, and (b) the covariance $C(t, s) = E(y(t) - Ey(t))(y(s) - Ey(s)), s, t \in \mathbb{R}$, show whether y is a wide sense stationary process.

Question 10.1 Solution

(a)

$$\begin{aligned} E[y(t)] &= \int_{-\infty}^{\infty} y(t) f_W(w)\, dw \\ &= \int_0^{2\pi} \sin(\theta t + w) \cdot \frac{1}{2\pi}\, dw \\ &= \frac{1}{2\pi} \left[\cos(\theta t + w)\right]_{w=2\pi}^{0} \\ &= \frac{1}{2\pi} \left[\cos(\theta t) - \cos(\theta t + 2\pi)\right] \\ &= 0 \end{aligned}$$

(b)

$$
\begin{aligned}
C(t,s) &= E(y(t) - Ey(t))(y(s) - E(y(s))) \\
&= E(y(t)y(s)) && \text{(substitute } Ey(t) \text{ and } Ey(s)) \\
&= \int_{-\infty}^{\infty} \sin(\theta t + w)\sin(\theta s + w) f_W(w)\,dw \\
&= \frac{1}{4\pi}\int_0^{2\pi} \cos(\theta t - \theta s) - \cos(\theta t + \theta s + 2w)\,dw && \text{(as } \sin A \sin B = \tfrac{\cos(A-B)-\cos(A+B)}{2}) \\
&= \frac{1}{4\pi}\left[\cos(\theta t - \theta s)w - \frac{1}{2}\sin(\theta t + \theta s + 2w)\right]_0^{2\pi} \\
&= \frac{1}{4\pi}\Big[2\pi\cos(\theta t - \theta s) - \frac{1}{2}\sin(t\theta t + \theta s - 4\pi) \\
&\qquad - 0 + \frac{1}{2}\sin(\theta t + \theta s)\Big] \\
&= \frac{1}{4\pi}\Big[2\pi\cos(\theta t - \theta s) - \frac{1}{2}\sin(t\theta t + \theta s) \\
&\qquad + \frac{1}{2}\sin(\theta t + \theta s)\Big] \\
&= \frac{1}{4\pi}2\pi\cos(\theta t - \theta s) \\
&= \frac{1}{2}\cos(\theta\tau) && \text{(where } \tau = t - s)
\end{aligned}
$$

Therefore $y(t)$ is w.s.s.

Question 10.2 (Random Bank Balances and the $ Fluctuates too!)

A bank client's income process I is a w.s.s. stochastic process with constant mean m^I and autocorrelation function $R^I(\tau)$, $\tau \in \mathbb{R}$, and the client's expenditure process K is a w.s.s. stochastic process with constant mean m^K and autocorrelation function $R^K(\tau)$, $\tau \in \mathbb{R}$. I and K are independent scalar processes. (We allow the income process to possibly take negative values (taxation!) and the expenditure process to possibly take positive values (refunds!).)

(a) Find the mean $m^B(t)$ and autocorrelation function $R^B(t+\tau, t)$, $t, \tau \in \mathbb{R}$, of the client's change of balance process $B(t) = I(t) - K(t)$, $t \in \mathbb{R}$. Is B a w.s.s. stochastic process ?

(b) Valued in a second (rather unstable!) currency, the client's change of balance process is given by $D(t) = s(t) \cdot B(t)$, $t \in \mathbb{R}$, where the exchange rate process $s(t)$, $t \in \mathbb{R}$, is a w.s.s. stochastic process which is independent of B and has mean 1 and autocorrelation function $R^s(\tau)$, $\tau \in \mathbb{R}$.

Show whether the autocorrelation function $R^D(t+\tau, t)$, $t, \tau \in \mathbb{R}$, is t-shift invariant.

Question 10.2 Solution

(a) We are given that $B(t) = I(t) - K(t), \quad t \in \mathbb{R}$. For $m^B(t)$:

$$\begin{aligned} m^B(t) &= E[B(t)] \\ &= E[I(t)] - E[K(t)] \\ &= m^I - m^K \end{aligned}$$

Now for $R^B(t+\tau, t)$:

$$\begin{aligned} R^B(t+\tau, t) &= E[B(t+\tau).B(t)] \\ &= E[I(t+\tau).I(t)] - E[I(t+\tau).K(t)] - E[K(t+\tau).I(t)] + E[K(t+\tau).K(t)] \\ &= R^I(\tau) - E[I(t+\tau)]E[K(t)] - E[K(t+\tau)]E[I(t)] + R^K(\tau), \end{aligned}$$

as $I(t)$ and $K(t)$ are indpendent.

$$\begin{aligned} R^B(t+\tau, t) &= R^I(\tau) - m^I.m^K - m^K.m^I + R^K(\tau) \\ &= R^I(\tau) + R^K(\tau) - 2.m^I.m^K \end{aligned}$$

Therefore $B(t)$ is w.s.s since its meanis constant and R is a function of τ.

(b) Check if the autocorrelation function $R^D(t+\tau,t)$, $t,\tau \in \mathbb{R}$, is t-shift invariant.

$$\begin{aligned} R^D(t+\tau,t) &= E[D(t+\tau)D(t)] \\ &= E[s(t+\tau).B(t+\tau).s(t).B(t)] \\ &= E[s(t+\tau).s(t)].E[B(t+\tau)B((t)], \end{aligned}$$

since $s(t)$ and $B(t)$ are independent. Then

$$R^D(t+\tau,t) = R^s(\tau).R^B(\tau)$$

Therefore $R^D(t+\tau,t)$ is t–shift invariant.

10.3 (Long Sequence of Filters in Series: Noise to Music?)

(i) A zero mean wide sense stationary scalar process X with covariance function $R_X(t)$, $t \in \mathbb{R}$, and spectral density $S_X(f)$, $f \in \mathbb{R}$, is passed through a linear filter (i.e. time invariant linear system) L with impulse response $\ell(t)$, $t \in \mathbb{R}$, and transfer function $L(f)$, $f \in \mathbb{R}$. The output process is denoted Y. Find an expression for the cross covariance $EY(t+\tau)X(t)$, $t, \tau \in \mathbb{R}$, in terms of an integral of the impulse response $\ell(\cdot)$ and the covariance function $R_X(\cdot)$ of the input process X.

(ii) Use the Wiener-Khinchin Theorem to give the spectral density $S_Y(f)$, $f \in \mathbb{R}$, of the process Y in terms of the transform of the impulse response $\ell(\cdot)$ and the spectral density of the process X.

(iii) If $L(f) = 0$, $|f| \geq W$, and $S_X(f) = 0$, $|f| \leq 2W$, what is the covariance function $R_Y(\tau)$, $\tau \in \mathbb{R}$? Explain your answer in terms of the frequency domain opeartion of the low pass filter L "matching" the input X; would L make a good suppressor of the process X if X were regarded as a noise process?

(iv) White noise N with spectral density 1 is passed into the first of a chain of n filters $L_1^n, L_2^n, \ldots, L_n^n$, where for each n the transfer function L_k^n is $(1 + \frac{\alpha 2\pi j f}{\sqrt{n}})^{-1}$, $f \in \mathbb{R}$, $1 \leq k \leq n$.

Give the spectral density $S_{Z_n}(f)$, $f \in \mathbb{R}$, of the process Z_n emitted at the output of the last filter L_n^n.

(v) Let $n \to \infty$ and give an analytic expression for the spectral density $S_{Z_\infty}(f)$, $f \in \mathbb{R}$.

(i)

$$\begin{aligned} E[Y(t+\tau)X(t)] &= E\left[\int_{-\infty}^{\infty} \ell(r).X(t+\tau-r)dr.X(t)\right] \\ &= \int_{-\infty}^{\infty} \ell(r).E[X(t+\tau-r).X(t)]dr \\ &= \int_{-\infty}^{\infty} \ell(r).R_X(\tau-r)dr \end{aligned}$$

(ii) $S_Y(f) = |L(f)|^2 S_X(f)$

(iii)

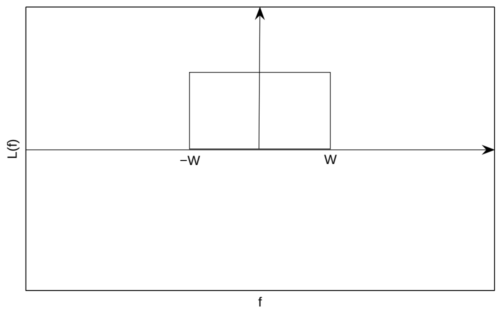

Figure 1: $L(f) = 0, \quad |f| \geq W$

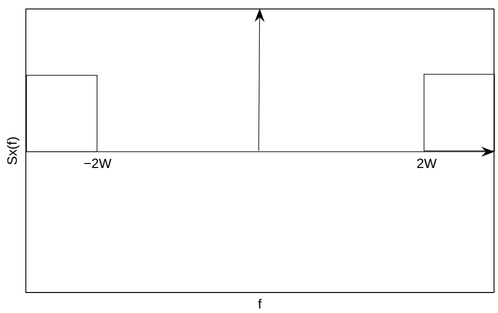

Figure 2: $S_X(f) = 0, \quad |f| \leq 2W$

$S_Y(f) = |L(f)|^2 S_X(f) = 0$. Hence the filter $L(f)$ is a good suppressor of the signal $X(t)$ since it completely filters out all the high frequencies X.

(iv) $S_{Z_n} = \left|\left(1 + \frac{j\alpha 2\pi f}{\sqrt{n}}\right)\right|^{2n} .S_X(f)$

(v)

$$\begin{aligned} S_{Z_n} &= \left| \frac{1}{1+\frac{j2\alpha\pi f}{\sqrt{n}}} \right|^{2n} .S_X(f) \\ &= \left| \frac{1}{1+\frac{4\alpha^2\pi^2 f^2}{n}} \right|^{n} .S_X(f) \\ &= \left| 1+\frac{4\alpha^2\pi^2 f^2}{n} \right|^{-n} .S_X(f) \end{aligned}$$

Using the fact that $\lim_{n\to\infty}(1+x/n)^n = \exp\{x\}$, it follows that

$$\lim_{n\to\infty} = S_{Z_n} = S_{Z_\infty} = \exp\{-4\alpha^2\pi^2 f^2\}.S_X(f)$$

10.4 (Radio City Contends with Noise in Signal)

(i) At Radio City a zero mean scalar wide sense stationary process X with correlation function $\{e^{-2\lambda|\tau|}, -\infty < \tau < \infty\}$, $\lambda > 0$, is passed through a linear system L with impulse response $\{e^{-\mu\tau}, 0 \leq \tau < \infty\}$, $\mu > 0$.

The output Y is disturbed by a zero mean scalar additive wide sense stationary noise process Z, where Z has correlation function $\{\sigma^2 e^{-2\gamma|\tau|}, -\infty < \tau < \infty\}$, $\gamma > 0$, and is independent of Y.

What is the spectral density of the resulting transmitted process $M = Y + Z$?

(ii) What is the signal to noise power ratio $\gamma_{\sigma^2} = \frac{S_M(f)}{S_Z(f)}$, $f \in \mathbb{R}$? What happens to γ_{σ^2} as the noise increases, i.e. $\sigma^2 \to \infty$, and decreases, i.e. $\sigma^2 \to 0$.

Question 10.4 Solution

(i) The autocorrelation of Y is given by

$$S_Y(f) = |H_L(f)|^2 S_X(f)$$

where $H_L(f)$ is the frequency response of L.

The autocorrelation of X is

$$R_X(\tau) = e^{-2\lambda|\tau|}$$

so its spectral density is

$$
\begin{aligned}
S_X(f) &= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|e^{-2\pi jf\tau}}\, d\tau \\
&= \int_{-\infty}^{0} e^{2\lambda\tau} e^{-j2\pi f\tau}\, d\tau + \int_{0}^{\infty} e^{-2\lambda\tau} e^{-j2\pi ft}\, d\tau \\
&= \left[\frac{e^{\tau(2\lambda - j2\pi f)}}{2\lambda - j2\pi f}\right]_{-\infty}^{0} + \left[\frac{-e^{\tau(2\lambda + j2\pi f)}}{2\lambda + j2\pi f}\right]_{0}^{\infty} \\
&= \frac{1}{2\lambda - j2\pi f} + \frac{1}{2\lambda + j2\pi f} \\
&= \frac{2\lambda + j2\pi f + 2\lambda - j2\pi f}{4\lambda^2 + 4\pi^2 f^2} \\
&= \frac{\lambda}{\lambda^2 + \pi^2 f^2};
\end{aligned}
$$

the frequency response of L is

$$
\begin{aligned}
H_L(f) &= \int_{0}^{\infty} e^{-\mu\tau} e^{-j2\pi f\tau}\, d\tau \\
&= \left[\frac{e^{-\tau(\mu + j2\pi f)}}{\mu + j2\pi f}\right]_{0}^{\infty} \\
&= \frac{1}{\mu + j2\pi f}
\end{aligned}
$$

and its squared magnitude is

$$
|H_L(f)|^2 = \frac{1}{\mu^2 + 4\pi^2 f^2}
$$

(The above expressions for $S_X(f)$ and $H_L(f)$ could also have been obtained from a table of Fourier transforms, instead of by integration.) Thus

$$
S_Y(f) = \frac{1}{(\lambda^2 + \pi^2 f^2)(\mu^2 + 4\pi^2 f^2)}
$$

Since Y and Z are independent,

$$
S_M(f) = S_Z(f) + S_Y(f);
$$

we have

$$
R_Z(f) = e^{-2\gamma|\tau|}
$$

$$
S_Z(f) = \frac{\gamma}{\gamma^2 + \pi^2 f^2} \qquad \text{(derived similarly to } S_Z(f)\text{)}
$$

and thus

$$S_M(f) = \frac{\lambda}{(\lambda^2 + \pi^2 f^2)(\mu^2 + 4\pi^2 f^2)} + \frac{\gamma}{\gamma^2 + \pi^2 f^2}.$$

(ii)