**ECSE 304-305 Assignment 10 (The Last Assignment) Fall 2008**

Return by noon, Tuesday, 2nd December

### 10.1 (Random Phase Shifted Signal)

The scalar sinusoidal signal $x(t) = sin(\theta t), t \in R$, passes through a channel $C$ where it is distorted by a constant (in time) random additive phase change between the input and the output of $C$. This produces the random signal

$$y(t) = sin(\theta t + w), \quad t \in R,$$

where the random disturbance $w$ is independent of time and is uniformly distributed on $[0,\ 2\pi]$, i.e. $w \sim U[0, 2\pi]$.

By explicitly calculating (a) the mean of the output process $Ey(t), t \in R$, and (b) the covariance $C(t,s) = E(y(t) - Ey(t))(y(s) - Ey(s)), s, t \in R$, show whether $y$ is a wide sense stationary process.

### 10.2 (Random Bank Balances and the $ Fluctuates too!)

A bank client's income process $I$ is a w.s.s. stochastic process with constant mean $m^I$ and autocorrelation function $R^I(\tau)$, $\tau \in R$, and the client's expenditure process $K$ is a w.s.s. stochastic process with constant mean $m^K$ and autocorrelation function $R^K(\tau)$, $\tau \in R$. $I$ and $K$ are independent scalar processes. (We allow the income process to possibly take negative values (taxation!) and the expenditure process to possibly take positive values (refunds!).)

(a) Find the mean $m^B(t)$ and autocorrelation function $R^B(t+\tau, t)$, $t, \tau \in R$, of the client's change of balance process $B(t) = I(t) - K(t)$, $t \in R$. Is $B$ a w.s.s. stochastic process ?

(b) Valued in a second (rather unstable!) currency, the client's change of balance process is given by $D(t) = s(t) \cdot B(t)$, $t \in R$, where the exchange rate process $s(t)$, $t \in R$, is a w.s.s. stochastic process which is independent of $B$ and has mean 1 and autocorrelation function $R^s(\tau)$, $\tau \in R$.

Show whether the autocorrelation function $R^D(t+\tau, t)$, $t, \tau \in R$, is $t$-shift invariant.

**10.3 (Long Sequence of Filters in Series: Noise to Music?)**

(i) A zero mean wide sense stationary scalar process $X$ with covariance function $R_X(t)$, $t \in R$, and spectral density $S_X(f)$, $f \in R$, is passed through a linear filter (i.e. time invariant linear system) $L$ with impulse response $l(t)$, $t \in R$, and transfer function $L(f)$, $f \in R$. The output process is denoted $Y$. Find an expression for the cross covariance $EY(t+\tau)X(t)$, $t, \tau \in R$, in terms of an integral of the impulse response $l(\cdot)$ and the covariance function $R_X(\cdot)$ of the input process $X$.

(ii) Use the Wiener-Khinchin Theorem to give the spectral density $S_Y(f)$, $f \in R$, of the process $Y$ in terms of the transform of the impulse response $l(\cdot)$ and the spectral density of the process $X$.

(iii) If $L(f) = 0$, $|f| \geq W$, and $S_X(f) = 0$, $|f| \leq 2W$, what is the covariance function $R_Y(\tau)$, $\tau \in R$? Explain your answer in terms of the frequency domain opeartion of the low pass filter $L$ "matching" the input $X$; would $L$ make a good suppressor of the process $X$ if $X$ were regarded as a noise process?

(iv) White noise $N$ with spectral density 1 is passed into the first of a chain of $n$ filters $L_1^n, L_2^n, \ldots, L_n^n$, where for each $n$ the transfer function $L_k^n$ is $(1 + \frac{\alpha 2\pi j f}{\sqrt{n}})^{-1}$, $f \in R$, $1 \leq k \leq n$.

Give the spectral density $S_{Z_n}(f)$, $f \in R$, of the process $Z_n$ emitted at the output of the last filter $L_n^n$.

(v) Let $n \to \infty$ and give an analytic expression for the spectral density $S_{Z_\infty}(f)$, $f \in R$.

**10.4 (Radio City Contends with Noise in Signal)**

(i) At Radio City a zero mean scalar wide sense stationary process $X$ with correlation function $\{e^{-2\lambda|\tau|}, -\infty < \tau < \infty\}$, $\lambda > 0$, is passed through a linear system $L$ with impulse response $\{e^{-\mu\tau}, 0 \leq \tau < \infty\}$, $\mu > 0$.

The output $Y$ is disturbed by a zero mean scalar additive wide sense stationary noise process $Z$, where $Z$ has correlation function $\{\sigma^2 e^{-2\gamma|\tau|}, -\infty < \tau < \infty\}$, $\gamma > 0$, and is independent of $Y$.

What is the spectral density of the resulting transmitted process $M = Y + Z$?

(ii) What is the signal to noise power ratio $\gamma_{\sigma^2} = \frac{S_M(f)}{S_Z(f)}$, $f \in R$? What happens to $\gamma_{\sigma^2}$ as the noise increases, i.e. $\sigma^2 \to \infty$, and decreases, i.e. $\sigma^2 \to 0$.