**ECSE 304-305B    Assignment 10    Winter 2007**

Return by noon, Wednesday, 11th April

**Marks for this assignment will not be counted towards the final grade. Material is within the syllabus. Return by due date in order to obtain an evaluation of work on the assignment.**

**Question 10.1 (Random Phase Shifted Signal)**

The scalar sinusoidal signal $x(t) = sin(\theta t), t \in R$, passes through a channel $C$ where it is distorted by a constant (in time) random additive phase change between the input and the output of $C$. This produces the random signal

$$y(t) = sin(\theta t + w), \quad t \in R,$$

where the random disturbance $w$ is independent of time and is uniformly distributed on $[0, 2\pi]$, i.e. $w \sim U[0, 2\pi]$.

By explicitly calculating (a) the mean of the output process $Ey(t), t \in R$, and (b) the covariance $C(t,s) = E(y(t) - Ey(t))(y(s) - Ey(s)), s, t \in R$, show whether $y$ is a wide sense stationary process.

**Question 10.1 Solution**

**(a)**

$$\begin{aligned} E[y(t)] &= \int_{-\infty}^{\infty} y(t) f_W(w)\, dw \\ &= \int_0^{2\pi} \sin(\theta t + w) \cdot \frac{1}{2\pi}\, dw \\ &= \frac{1}{2\pi} \left[\cos(\theta t + w)\right]_{w=2\pi}^{0} \\ &= \frac{1}{2\pi} \left[\cos(\theta t) - \cos(\theta t + 2\pi)\right] \\ &= 0 \end{aligned}$$

**(b)**

$$\begin{aligned}
C(t,s) &= E(y(t)-Ey(t))(y(s)-E(y(s))) \\
&= E(y(t)y(s)) && \text{(substitute } Ey(t) \text{ and } Ey(s)) \\
&= \int_{-\infty}^{\infty} \sin(\theta t + w)\sin(\theta s + w) f_W(w)\, dw \\
&= \frac{1}{4\pi}\int_0^{2\pi} \cos(\theta t - \theta s) - \cos(\theta t + \theta s + 2w)\, dw && (\text{as } \sin A \sin B = \tfrac{\cos(A-B)-\cos(A+B)}{2}) \\
&= \frac{1}{4\pi}\left[\cos(\theta t - \theta s)w - \frac{1}{2}\sin(\theta t + \theta s + 2w)\right]_0^{2\pi} \\
&= \frac{1}{4\pi}\Big[2\pi\cos(\theta t - \theta s) - \frac{1}{2}\sin(t\theta t + \theta s - 4\pi) \\
&\qquad - 0 + \frac{1}{2}\sin(\theta t + \theta s)\Big] \\
&= \frac{1}{4\pi}\Big[2\pi\cos(\theta t - \theta s) - \frac{1}{2}\sin(t\theta t + \theta s) \\
&\qquad + \frac{1}{2}\sin(\theta t + \theta s)\Big] \\
&= \frac{1}{4\pi}2\pi\cos(\theta t - \theta s) \\
&= \frac{1}{2}\cos(\theta\tau) && (\text{where } \tau = t - s)
\end{aligned}$$

Therefore $y(t)$ is WSS.

**Question 10.2**

At NoiseCity Radio a scalar wide sense stationary process $X$ with correlation function $\{e^{-2\lambda|\tau|}, -\infty < \tau < \infty\}$, $\lambda > 0$, is passed through a linear system $L$ with impulse response $\{e^{-\mu\tau}, 0 \le \tau < \infty\}$, $\mu > 0$.

The output $Y$ is added to a scalar wide sense stationary process $Z$, where $Z$ has correlation function $\{e^{-2\gamma|\tau|}, -\infty < \tau < \infty\}$, $\gamma > 0$, and is independent of $Y$.

What is the spectral density of the process $W = Y + Z$?

**Question 10.2 Solution**

The autocorrelation of $Y$ is given by

$$S_Y(f) = |H_L(f)|^2 S_X(f)$$

where $H_L(f)$ is the frequency response of $L$.

The autocorrelation of $X$ is

$$R_X(\tau) = e^{-2\lambda|\tau|}$$

so its spectral density is

$$\begin{aligned}
S_X(f) &= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|e^{-2\pi jf\tau}}\, d\tau \\
&= \int_{-\infty}^{0} e^{2\lambda\tau}e^{-j2\pi f\tau}\, d\tau + \int_{0}^{\infty} e^{-2\lambda\tau}e^{-j2\pi ft}\, d\tau \\
&= \left[\frac{e^{\tau(2\lambda-j2\pi f)}}{2\lambda - j2\pi f}\right]_{-\infty}^{0} + \left[\frac{-e^{\tau(2\lambda+j2\pi f)}}{2\lambda + j2\pi f}\right]_{0}^{\infty} \\
&= \frac{1}{2\lambda - j2\pi f} + \frac{1}{2\lambda + j2\pi f} \\
&= \frac{2\lambda + j2\pi f + 2\lambda - j2\pi f}{4\lambda^2 + 4\pi^2 f^2} \\
&= \frac{\lambda}{\lambda^2 + \pi^2 f^2};
\end{aligned}$$

the frequency response of $L$ is

$$\begin{aligned}
H_L(f) &= \int_{0}^{\infty} e^{-\mu\tau}e^{-j2\pi f\tau}\, d\tau \\
&= \left[\frac{e^{-\tau(\mu+j2\pi f)}}{\mu + j2\pi f}\right]_{0}^{\infty} \\
&= \frac{1}{\mu + j2\pi f}
\end{aligned}$$

and its squared magnitude is

$$|H_L(f)|^2 = \frac{1}{\mu^2 + 4\pi^2 f^2}$$

(The above expressions for $S_X(f)$ and $H_L(f)$ could also have been obtained from a table of Fourier transforms, instead of by integration.) Thus

$$S_Y(f) = \frac{1}{(\lambda^2 + \pi^2 f^2)(\mu^2 + 4\pi^2 f^2)}$$

Since $Y$ and $Z$ are independent,

$$S_W(f) = S_Z(f) + S_Y(f);$$

we have

$$R_Z(f) = e^{-2\gamma|\tau|}$$

$$S_Z(f) = \frac{\gamma}{\gamma^2 + \pi^2 f^2} \qquad \text{(derived similarly to } S_Z(f))$$

and thus

$$S_W(f) = \frac{\lambda}{(\lambda^2 + \pi^2 f^2)\,(\mu^2 + 4\pi^2 f^2)} + \frac{\gamma}{\gamma^2 + \pi^2 f^2}.$$

## Question 10.3

Let $S_X(f)$, $f \in R$ be the spectral density of the zero mean wide sense stationary scalar process $X$ with autocorrelation function $R(\tau)$, $\tau \in R$.

(a) It is required to synthesize a stochastic signal $P$ with spectral density $S_P(f) = \frac{1}{\alpha^2+4\pi^2 f^2}$, $f \in R$, $\alpha > 0$, by passing white noise $W$ with spectral density $S_W(f) = 1, f \in R$, through a stable filter $H(f)$, $f \in R$.

(i) Find a stable filter $H$ (in the frequency domain) whose output is $P$.

(ii) Find the impulse response of $H$ on $t \geq 0$.

(b) $P$ is passed through a linear system $L$ where $L(f) = \frac{\alpha-2\pi jf}{\gamma+2\pi jf}$, $f \in R$, $\gamma > 0$. Give the spectral density of the output $Z$ of $L$ in its simplest form.

(c) The process $Y = Z+N$ is formed by summing $Y(t)$ and the white noise process $N(t)$ at each $t \in R$, where the processes $Y$ and $N$ are independent and $N$ has spectral density $S_W(f) = 2, f \in R$. What is the spectral density of $Y$?

## Question 10.3 Solution

**(a.i)** The spectral density of $P$ is given by

$$S_P(f) = |H(f)|^2 S_W(f)$$

where $H(f)$ is the frequency response of the desired filter. Thus,

$$|H(f)|^2 = \frac{S_P(f)}{S_W(f)} = \frac{1}{\alpha^2 + 4\pi^2 f^2}$$

A stable filter $H$ such that $|H(f)|^2 = \dfrac{1}{\alpha^2 + 4\pi^2 f^2}$ is $H(f) = \dfrac{1}{\alpha + 2\pi f}$

**(a.ii)** The time domain impulse response is:

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

**(b)**

$$S_Z(f) = |L(f)|^2 S_P(f) = L(f)L(f)^* S_P(f) = \left(\frac{\alpha - 2\pi jf}{\gamma + 2\pi jf}\right)\left(\frac{\alpha + 2\pi jf}{\gamma - 2\pi jf}\right)\left(\frac{1}{\alpha^2 + 4\pi^2 f^2}\right) = \frac{1}{\gamma^2 + 4\pi^2 f^2}$$

**(c)**

$$S_Y(f) = S_X(f) + S_N(f) = \frac{1}{\gamma^2 + 4\pi^2 f^2} + 2$$