ECSE 304-305 Solutions to Assignment 9 Fall 2008

Solution 9.1

(a)

If $\rho = 0$, then

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{(x-m_1)^2}{2\sigma_1}\right\} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left\{-\frac{(y-m_2)^2}{2\sigma_2}\right\} \\ &= f_X(x) \cdot f_Y(y). \end{aligned}$$

Hence, X and Y are independent.

(b)

$$\begin{aligned} P[XY > 0] &= P[X > 0, Y > 0] \cup P[X < 0, Y < 0] \\ &= P[X > 0, Y > 0] + P[X < 0, Y < 0] + \underbrace{P[X > 0, Y > 0] \cap P[X < 0, Y < 0]}_{= \emptyset}. \end{aligned}$$

Since $m_1 = m_2 = 0$, $\rho = 0$, X and Y are zero mean, independent Gaussian random variables. Hence

$$\begin{aligned} P[X > 0, Y > 0] &= P[X > 0]P[Y > 0] = 1/4 \\ P[X < 0, Y < 0] &= P[X < 0]P[Y < 0] = 1/4, \end{aligned}$$

and $P[XY > 0] = 1/4 + 1/4 = 1/2$.

(c)

$$\begin{aligned} R_{1,1} &:= E(X - m_1)^2 = \sigma_1^2 \\ R_{2,2} &:= E(Y - m_2)^2 = \sigma_2^2 \\ R_{1,2} &:= E(X - m_1)(Y - m_2) = EXY - m_1 m_2 \end{aligned}$$

since $EXY = \rho\sigma_1\sigma_2 + m_1 m_2$,

$$R_{1,2} = \rho\sigma_1\sigma_2 = R_{2,1}.$$

then for $n = 2$,

$$\begin{aligned}
(2\pi)^{n/2}|\det R|^{1/2} &= 2\pi \left|\det \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right|^{1/2} \\
&= 2\pi \left(\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2\right)^{1/2} \\
&= 2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}.
\end{aligned}$$

Solution 9.2

(i)

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} exp\left\{\frac{-1}{2}[x,y]\,\Sigma_{X,Y}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right\} \\
&= \frac{1}{2\sqrt{3}\pi} e^{\frac{-1}{3}\left(x^2-xy+y^2\right)}
\end{aligned}$$

(ii) from 1(c)

$$2\pi|R|^{\frac{1}{2}} = 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}$$

where $\sigma_1^2 = 2$, $\sigma_2^2 = 2$ and $|R| = 3$. This yields $\rho = 1/2$.

(iii)

$$\begin{aligned}
\mu_{U,V} &= \left[\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right] \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} E\begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\end{aligned}$$

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$\begin{aligned}
cov(U,V) &= cov\left(\frac{1}{\sqrt{2}}AX, \frac{1}{\sqrt{2}}A^TY\right) \\
&= \frac{1}{2}Acov(X,Y)A^T \\
&= \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

(iv) $\rho_{U,V} = 0$

(v)

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi\sigma_u\sigma_v} exp\left[\frac{-1}{2}\left(\frac{(u-\mu_u)^2}{\sigma_u^2} + \frac{(v-\mu_v)^2}{\sigma_v^2}\right)\right] \\ &= \frac{1}{2\pi\sqrt{3}} exp\left[\frac{-(u-2)^2}{6} - \frac{(v+1)^2}{2}\right] \end{aligned}$$

Solution 9.3

(a)

$$\begin{aligned} P(N_{T_3} = n_3, N_{T_2} = n_2, N_{T_1} = n_1) &= P(N_{T_3} = n_3 | N_{T_2} = n_2, N_{T_1} = n_1) P(N_{T_2} = n_2, N_{T_1} = n_1) \\ &= P(N_{T_3} = n_3 | N_{T_2} = n_2, N_{T_1} = n_1) P(N_{T_2} = n_2 | N_{T_1} = n_1) P(N_{T_1} = n_1) \\ &= P(N_{T_3 - T_2} = n_3 - n_2) P(N_{T_2 - T_1} = n_2 - n_1) P(N_{T_1} = n_1) \\ &= \frac{\lambda_{n_3}(T_3 - T_2)^{n_3 - n_2}(T_2 - T_1)^{n_2 - n_1}(T_1)^{n_1} e^{-\lambda T_3}}{(n_3 - n_2)!(n_2 - n_1)!(n_1)!} \end{aligned}$$

(b)

$$\begin{aligned} P(N_{T_1} = n_1 | N_{T_2} = n_2) &= \frac{\lambda^{n_2 - n_1}(T_2 - T_1)^{n_2 - n_1} e^{-\lambda(T_2 - T_1)(n_2 - n_1!)}}{(n_2 - n_1)!} \frac{\lambda^{n_1} T_1^{n_1} e^{-\lambda T_1}}{(n_1!)} \frac{n_2!}{(\lambda T_2)^{n_2} e^{-\lambda T_2}} \\ &= \binom{n_2}{n_1}((\frac{T_1}{T_2})^{n_1}(1 - (\frac{T_1}{T_2}))^{n_2 - n_1} \end{aligned}$$

(c)

$$P = \int_0^T \sum_{l=0}^{L} \frac{\gamma^{L-l}(T-t)^{L-l} e^{-\gamma(T-t)}}{(L-l)!} \frac{\lambda^l t^l e^{-\lambda t}}{l!} \mu e^{-\mu t} dt$$

(d)

$$P = \int_0^T \frac{\lambda T^L}{L!} e^{-\lambda T} \mu e^{-\mu t}$$