ECSE 304-305B Assignment 9 Fall 2008

Return by noon, Friday, 21st November

9.1 The general form of the joint probability density function for two Gaussian random variables is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\{\frac{-1}{2(1-\rho^2)}\Big[\Big(\frac{x-m_1}{\sigma_1}\Big)^2 - 2\rho\Big(\frac{x-m_1}{\sigma_1}\Big)\Big(\frac{y-m_2}{\sigma_2}\Big) + \Big(\frac{y-m_2}{\sigma_2}\Big)^2\Big]\}$$

(a) Show that if $\rho = 0$, then X and Y are independent.

(b) Suppose $\rho = 0$ and $m_1 = m_2 = 0$. Find $P[\{X > 0\} \cup \{Y = 0\} \cup \{X > 0 \cap \{XY < 0\}]$. Your answer should not involve any integration.

(c) Write the general density $f_{X,Y}(x,y)$ above in the alternative form for a mulitivariable Gaussian density :

$$p(x,y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{|R|^{1/2}} \exp\left\{ -\frac{1}{2}[x - m_1, y - m_2] R^{-1} \begin{bmatrix} x - m_1 \\ y - m_2 \end{bmatrix} \right\}$$

where $n = 2, (x,y) \in \mathbf{R}^2$ and R is a 2×2 matrix with entries $R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}$.

Find the values of the (scalars) $R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}$ in terms of the parameters in the density function above, and then show that $(2\pi)^{\frac{n}{2}}|detR|^{\frac{1}{2}} = 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}$.

9.2 The bivariate Gaussian random variable $\begin{bmatrix} X \\ Y \end{bmatrix}$ has the mean and covariance

$$\mu_{X,Y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Sigma_{X,Y} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(i) Give the probability density $f_{X,Y}(x,y)$, $(x,y) \in R^2$, of $\begin{bmatrix} X \\ Y \end{bmatrix}$ with the argument of the exponential written as a *scalar* function of (x,y).

(ii) Find the correlation coefficient $\rho_{X,Y}$.

(iii) Find the mean $\mu_{U,V}$ and covariance $\Sigma_{U,V}$ of $\begin{bmatrix} U \\ V \end{bmatrix}$ when

$$\begin{bmatrix} U \\ V \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(iv) Give the correlation coefficient $\rho_{U,V}$.

(v) Give the probability density $f_{U,V}(u,v)$, $(u,v) \in R^2$, of $\begin{bmatrix} U \\ V \end{bmatrix}$ in terms of the values of $\mu_{U,V}$ and $\Sigma_{U,V}$ found in (iii).

9.3

A manufacturing system is governed by a Poisson counting process $N = \{N_t; 0 \le t < \infty\}$; the process N has a rate parameter $\lambda > 0$ and it starts at $t = 0$ with the value $N_0 = 0$ with probability 1.

(a) Use the independent increment property of the Poisson process to give a formu la for the joint probability

$$P(N_{T_3} = n_3, N_{T_2} = n_2, N_{T_1} = n_1),$$

where T_1, T_2, T_3 are fixed times $T_1 \le T_2 \le T_3$, and $n_1 \le n_2 \le n_3$.

(b) If $N_{T_2} = n_2$ events are observed over $[0, T_2]$, find the conditi onal probability that $N_{T_1} = n_1$ events are observed over $[0, T_1]$. Does this conditional probability depend upon λ?

(c) An exponential process E with rate $\mu > 0$ runs in parallel to N. When an event happens in the E process the counting process N changes to a second counting process $M = \{M_t; 0 \leq t < \infty\}$ with rate paramete r $\gamma > 0$. All three processes are independent. (α) Give a formula for the probability of the event: The total number of counting events is equal to L and there is one switching event E in a given interval of time $[0, T]$.

Do this by (i) for some fixed $t \in [0, T]$, summing over all possible combinations of l counting events for N on $[0, t]$, and then $L - l$ counting events for M on $[t, T]$, which could give rise to a total of L counting events in the period $[0, T]$, where the process switch caused by E takes place in a period $[t, t + dt]$ with probability $\mu e^{-\mu t} dt$. Then, (ii) by writing the final formual as an integral over all such t.

Give all probabilities in (i) explicitly in terms of Possion distributions, but do not evaluate the integral.

(β) In case $\lambda = \gamma$ simplify the formula in (α), and explain the result in terms of the standard single Poisson process case where there is no switching process E.