ECSE 304-305B Assignment 9 Winter 2007

Return by noon, Thursday, 5th April

9.1 The general form of the joint probability density function for two Gaussian random variables is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\{\frac{-1}{2(1-\rho^2)}\Big[\Big(\frac{x-m_1}{\sigma_1}\Big)^2 - 2\rho\Big(\frac{x-m_1}{\sigma_1}\Big)\Big(\frac{y-m_2}{\sigma_2}\Big) + \Big(\frac{y-m_2}{\sigma_2}\Big)^2\Big]\}$$

(a) Show that if $\rho = 0$, then X and Y are independent.

(b) Suppose $\rho = 0$ and $m_1 = m_2 = 0$. Find $P[\{X > 0\} \cup \{Y = 0\} \cup \{X > 0 \cap \{XY < 0\}]$. Your answer should not involve any integration.

(c) Write the general density $f_{X,Y}(x,y)$ above in the alternative form for a mulitivariable Gaussian density :

$$p(x,y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{|R|^{1/2}} \exp\left\{ -\frac{1}{2}[x-m_1, y-m_2] R^{-1} \begin{bmatrix} x-m_1 \\ y-m_2 \end{bmatrix} \right\}$$

where $n = 2, (x,y) \in \mathbf{R}^2$ and R is a 2×2 matrix with entries $R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}$.

Find the values of the (scalars) $R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}$ and show that $(2\pi)^{\frac{n}{2}} |detR|^{\frac{1}{2}} = 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}$.

9.2 The bivariate Gaussian random variable $\begin{bmatrix} X \\ Y \end{bmatrix}$ has the mean and covariance

$$\mu_{X,Y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Sigma_{X,Y} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(i) Give the probability density $f_{X,Y}(x,y)$, $(x,y) \in R^2$, of $\begin{bmatrix} X \\ Y \end{bmatrix}$ with the argument of the exponential written as a *scalar* function of (x,y).

(ii) Find the correlation coefficient $\rho_{X,Y}$.

(iii) Find the mean $\mu_{U,V}$ and covariance $\Sigma_{U,V}$ of $\begin{bmatrix} U \\ V \end{bmatrix}$ when

$$\begin{bmatrix} U \\ V \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(iv) Give the correlation coefficient $\rho_{U,V}$.

(v) Give the probability density $f_{U,V}(u,v)$, $(u,v) \in R^2$, of $\begin{bmatrix} U \\ V \end{bmatrix}$ in terms of the values of $\mu_{U,V}$ and $\Sigma_{U,V}$ found in (iii).

9.3 Let the bivariate random variable (Z, W) have the density

$$\frac{1}{2\pi} \exp\{-(\frac{z^2 - 2zw + 2w^2}{2})\} \qquad (z, w) \in R^2$$

Let (Z, W) be mapped to (X, Y) by the transformation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2Z - 3W \\ 4W \end{bmatrix}.$$

Give the density $f_{X,Y}(x,y)$, $(x,y) \in R^2$, in the form of the density $p(x,y)$ in Question 9.1 above; clearly specify the values of R, $\det R$, and m_1, m_2, in the notation of that question.