ECSE 304-305B Assignment 7 Winter 2007

Return by 12.00 pm, 12th March

7.1

Consider a sequence of independent random variables $\{X_k; 1 \leq k\}$ such that X_k is uniformly distributed on the interval $[-k, k]$. Define:

$$Y_n = \sum_{k=1}^{n} X_k, \qquad Z_n = \sum_{k=1}^{n} (-1)^k X_k$$

Show by the use of characteristic functions that Y_n and Z_n have identical distributions for all $\{n; 1 \leq n\}$.

7.2

(i) Let the exponentially distributed random variable X with parameter $\lambda > 0$ model the waiting time until the random instant at which an event occurs:

$$P(X \leq t) \quad = \quad 1 - e^{-\lambda t} \qquad\qquad t \in R_+$$

Show that X possesses the memoryless property:

$$P(\ X\ >\ t + h \mid X\ >\ t\ )\ =\ P(\ X\ >\ h\ ).$$

This may be interpreted as the waiting process restarting from zero at any given time.

[Hence, if the occurrence of an event is exponentially distributed, the fact that one has waited t seconds for it to happen has no influence on the probability whether you will see the event occur in the next h seconds. (This is viewed as bad by someone in an exponential bus queue; one's investment in waiting is of no value.)]

(ii) Give the characteristic function of the exponential waiting time distribution on $[0, \infty)$ with parameter $\lambda > 0$.

A traveller at Trudeau International Airport must wait in two queues in series: first, the traveller must wait at the Check-in queue for his or her airline; this has an exponentially distributed waiting time T_λ, with parameter $\lambda > 0$; second, the traveller must wait in a queue in the Security Zone with an exponentially distributed waiting time T_μ, with parameter $\mu > 0$.

It is assumed that T_λ and T_μ are independent random variables.

(iii) What is the characteristic function of the total waiting time $T_\lambda + T_\mu$?

(iv) Find the characteristic function of $2T_\lambda$.

(v) By use of characteristic functions, or otherwise, show whether the density of $T_\lambda + T_\mu$ with $\mu = \lambda$ is the same as that of $2T_\lambda$.

(vi) Find the second moment of $T_\lambda + T_\mu$.

7.3

For a random variable X with a probability density function $f_X(\cdot)$, let $Y = g(X)$, and consider the four cases:

(a) $g(x) = -x$, where X is uniformly distributed on $[-1, 1]$,

(b) $g(x) = x^3$, where X is uniformly distributed on $[1, 4]$,

(c) $g(x) = 2|x|$, where X has the Gaussian density $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} exp^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$,

(d) $g(x) = -|x + 2|$, where X has the Gaussian density $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} exp^{-\frac{1}{2}\frac{(x-1)^2}{\sigma^2}}$.

Find the formula for the probability density $f_Y(\cdot)$ of Y in each case at values of y for which $\frac{dy}{dx}$ exists and $\frac{dy}{dx} \neq 0$.

7.4

Assume each of the independent identically distributed scalar random variables $X_i, 1 \leq i < \infty$, has mean 0 and variance $\sigma^2 = 4$. For $\alpha > 0$, consider the probability of the event :

$$A = \{-\alpha \leq \tfrac{1}{n} \textstyle\sum_{i=1}^{n} X_i \leq \alpha\}$$

(i) Use the Central Limit Theorem, together with the notation $\Phi(x), x \in R$, for the distribution function of a normally distributed $N(0, 1)$ random variable, to give a formula for an approximation to the probability that the average $Z_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ lies in the interval $[-\alpha, \alpha]$.

(ii) Let $\alpha = 1$. Use the CLT based formula to find the smallest value of n for which the probability of A is at least: (a) 0.95 and (b) 0.9786. (You may use the fact that for the Gaussian distribution $\Phi(-x) = 1 - \Phi(x), x \in R$, and may use any standard Gaussian distribution table; for instance in the course text this is given on page SG 632.)

7.5

The random variable X has the Binomial distribution $B(N, \frac{1}{2})$, i.e. it is the sum of N independent Bernoulli $\{+1, -1\}$ valued random variables $\{Y_k; 1 \leq k \leq N\}$ each of which satisfies $P(Y_k = +1) = P(Y_k = -1) = \frac{1}{2}$.

(a) find EX,

(b) show whether $E(X^2)$ has the value N or $\frac{1}{2}N$,

(c) check you answer in (b) in the case $N = 2$,

(d) use the Chebychev inequality to estimate $P(|X - EX| > N)$.

Hint: The Moment Theorem and the characteristic function $Ee^{iX\omega}$ may be used in parts (a) and (b) if you wish.