Problem Set 12

Solutions

ProblemSet_12: 1

11-2

Eq. (11-2): $\bar{E} = -\bar{\nabla} V - jw\bar{A} = \bar{a}_R E_R + \bar{a}_\theta E_\theta + \bar{a}_\phi E_\phi$

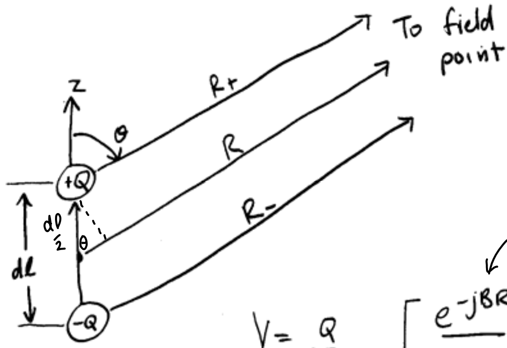
$$E_R = -\frac{\partial V}{\partial R} - jwA_R$$

$$E_\theta = -\frac{\partial V}{R\partial \theta} - jwA_\theta$$

$$E_\phi = -\frac{\partial V}{R\sin\theta\partial \phi} - jwA_\phi$$

The expressions of A_R, A_θ, and A_ϕ are given in Eqns. (11-14a, b and c)

ProblemSet_12: 2



k

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{e^{-j\beta R_+}}{R_+} - \frac{e^{-j\beta R_-}}{R_-}\right]$$

$$R_+ \cong R - \tfrac{1}{2}\, dl \cos\theta$$

$$R_- \cong R + \tfrac{1}{2}\, dl \cos\theta$$

ProblemSet_12: 3

$$Q = \frac{I}{j\omega}, \quad (dl)^2 << R^2$$

$$\frac{1}{R^+} \cong \frac{1}{R}\left(1 - \frac{1}{2}\frac{dl}{R}\cos\theta\right)^{-1}$$

$$\cong \frac{1}{R}\left(1 + \frac{dl}{2R}\cos\theta\right)$$

$$V \cong \frac{I e^{-j\beta R}}{4\pi\epsilon_0 j\omega}\frac{1}{R^2}\Big[\left(R + \frac{dl}{2}\cos\theta\right)e^{j\beta(dl\cos\theta)/2} - \left(R - \frac{dl}{2}\cos\theta\right)e^{-j\beta(dl\cos\theta)/2}\Big]$$

$$= \frac{I e^{-j\beta R}}{4\pi\epsilon_0 j\omega R^2}\Big[2jR\sin\left(\frac{\beta\, dl\cos\theta}{2}\right) + 2\left(\frac{dl}{2}\cos\theta\right)\cos\left(\frac{\beta\, dl \cos\theta}{2}\right)\Big]$$

ProblemSet_12: 4

$$V \simeq \frac{I e^{-j\beta R}}{4\pi\epsilon_0 j\omega R^2}\left[2jR\left(\frac{\beta dl \cos\theta}{2}\right) + dl\cos\theta\right]$$

$$= \frac{I dl \cos\theta}{4\pi R^2}\,\eta_0\left(R + \frac{1}{j\beta}\right)e^{-j\beta R}$$

Using A_R, A_θ, A_ϕ and V in E_R, E_θ, and E_ϕ, we obtain the same results as given in Eqns. (11-16 a,b,c).

ProblemSet_12: 5

$$\underline{A} = \mu I \frac{dl\, e^{-jkR}}{4\pi R}\left(\cos\theta\, \underline{a}_R - \sin\theta\, \underline{a}_\theta\right) \qquad V = \eta \frac{I dl}{4\pi R^2} e^{-jkR}\left(R + \frac{1}{jk}\right)\cos\theta$$

$$E_R = -\frac{\partial V}{\partial R} - j\omega A_R =$$

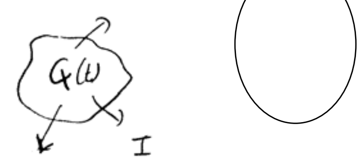
$$= -\frac{I dl}{4\pi}\eta k^2\, 2\cos\theta\left[\frac{1}{(jkR)^3} + \frac{1}{(jkR)^2}\right]e^{-jkR}$$

ProblemSet_12: 6

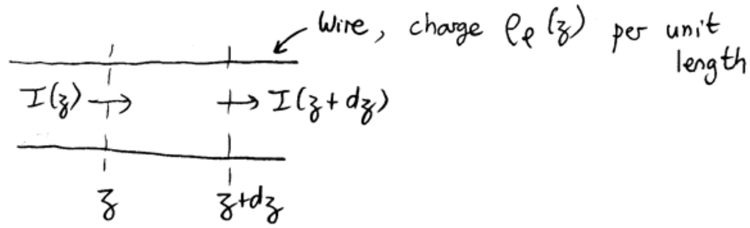
11-5

$$I = -\frac{dQ}{dt}$$ charge in volume

current out of volume



$$I = -j\omega Q \quad \text{in time-harmonic case}$$



ProblemSet_12: 7

So $\quad I(z+dz) - I(z) = -j\omega\,(\rho_\ell\, dz)$

$$\Rightarrow \quad \frac{dI}{dz} = -j\omega \rho_\ell \quad \Rightarrow \quad \rho_\ell = \frac{j}{\omega}\frac{dI}{dz}$$

(a) $I(z) = I_o \cos\beta z \quad \Rightarrow \quad \rho_\ell = -\frac{j}{\omega} I_o \beta \sin\beta z = -j\frac{I_o}{c}\sin\beta z$

(b) $I(z) = I_o\left(1 - \frac{4}{\lambda}|z|\right)$

$$\Rightarrow \quad \rho_\ell = \begin{cases} -\frac{j}{\omega} I_o \frac{4}{\lambda} = -\frac{2jI_o}{\pi c} & z > 0 \\ +j\frac{I_o}{\omega}\frac{4}{\lambda} = +\frac{2jI_o}{\pi c} & z < 0 \end{cases}$$

ProblemSet_12: 8

11-6

$\lambda = \frac{c}{f} = \frac{3\times10^8}{10^6} = 300$ m

Then $\frac{dl}{\lambda} = \frac{15}{300} \ll 1$ So: Hertzian dipole

Omit – not covered

a) From (11-44): $R_r = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 1.97\,\Omega$

b) From (11-48), resistance of wire $R_l = R_s \times \frac{dl}{2\pi a}$

where $R_s = \sqrt{\frac{\pi f \mu_0}{\sigma}} = \sqrt{\frac{\pi\times10^6\times4\pi\times10^{-7}}{5.8\times10^7}} = 2.61\times10^{-4}\,\Omega$

ProblemSet_12: 9

Then $R_l = 2.61\times10^{-4} \times \frac{15}{2\pi\times2\times10^{-2}} = 0.031\,\Omega$

From (11-47), efficiency $\eta_r = \frac{R_r}{R_r+R_l} = 98.5\ \%$

c) From (11-43), Power radiated $P_r = \frac{|I|^2}{12\pi} dl^2 \eta_0 \beta^2$

From (11-19b), $|E_\theta|_{max} = |I|\frac{dl}{4\pi}\frac{1}{R}\eta_0\beta$ (in $\theta = \frac{\pi}{2}$ direction)

Combining: $|E_\theta|_{max} = \sqrt{\frac{12\pi P_r}{dl^2\eta_0\beta^2}}\ \frac{dl}{4\pi}\frac{1}{R}\eta_0\beta = \frac{1}{R}\sqrt{90P_r}$

or see next page

$= \frac{1}{20\times10^{+3}}\sqrt{90\times1.6\times10^3} = 19.0$ mV/m

ProblemSet_12: 10

(c) $P_r = \frac{4}{3} \pi \eta_0 |c|^2 \qquad \left(c = j \frac{I\,dl}{4\pi} k \right)$

$$E_\theta = \eta_0 c \sin\theta \frac{e^{-jkR}}{R}$$

$$\Rightarrow \max |E_\theta| = \frac{\eta_0 |c|}{R} = \frac{\eta_0}{R} \sqrt{\frac{3P_r}{4\pi\eta_0}} = \frac{1}{R} \sqrt{\frac{3 \cdot 120\pi \; P_r}{4\pi}}$$

$$= \frac{1}{R} \sqrt{90 P_r}$$

$R = 20\ \text{km}$
$P_r = 1.6\ \text{kW}$ $\qquad \max |E_\theta| = 19\ \text{mV m}^{-1}$

ProblemSet_12: 11

11-11 $\quad P_R = \oint \bar{P}_{AV} \cdot d\bar{s} = \frac{1}{2} \,\text{Re} \int_0^{2\pi} \int_0^{\pi} E_\theta H_\phi^* R^2 \sin\theta \, d\theta \, d\phi$

$$= \frac{(I\,dl)^2}{16\pi} \beta^4 R^2 \eta_0 \,\text{Re} \left\{ \left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} \right] \cdot \left[-\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} - \frac{1}{(j\beta R)^3} \right] \right\} \cdot \int_0^{\pi} \sin^3\theta \, d\theta$$

$P_r = \frac{I^2}{2} \left[80\pi^2 \left(\frac{dl}{\lambda} \right)^2 \right]$, which is the same as Eq. 11-43.

ProblemSet_12: 12

$$H_\varphi = -\frac{I\,d\ell}{4\pi} k^2 \sin\theta \left[\frac{1}{(jkR)^2} + \frac{1}{jkR} \right] e^{-jkR}$$

$$E_\theta = -\frac{I\,d\ell}{4\pi} \eta_0 k^2 \sin\theta \left[\frac{1}{(jkR)^3} + \frac{1}{(jkR)^2} + \frac{1}{jkR} \right] e^{-jkR}$$

$$H_\varphi E_\theta^* = \frac{|I|^2 d\ell^2}{(4\pi)^2} \eta_0 k^4 \sin^2\theta \left[\frac{1}{(jkR)^2} + \frac{1}{jkR} \right] \times$$

$$\left[\frac{1}{(jkR)^3} + \frac{1}{(jkR)^2} + \frac{1}{jkR} \right]^*$$

ProblemSet_12: 13