Problem Set 9

Solutions

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1.

(9-10)

For 2-wire transmission line:

a) $Z_0 = \sqrt{\frac{L}{C}}$

$L = \frac{\mu}{\pi} \cosh^{-1}\left(\frac{D}{2a}\right)$

$C = \frac{\pi \epsilon}{\cosh^{-1}(D/2a)}$

$$Z_0 = \sqrt{\frac{\frac{\mu}{\pi}\cosh^{-1}(D/2a)}{\pi\epsilon / \cosh^{-1}(D/2a)}} = \sqrt{\frac{\mu}{\pi^2 \epsilon}\left(\cosh^{-1}(D/2a)\right)^2}$$

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$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\cosh^{-1}\left(\frac{D}{2a}\right) = 300\,\Omega$$

$$= \frac{120}{\sqrt{\epsilon_r}} \ln\left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right] = 300$$

$$\frac{D}{2a} = 21.27 \longrightarrow \boxed{D = 25.5\times 10^{-3}\ \text{m}}$$

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b) For a coax,

$$L = \frac{\mu}{2\pi}\ln\frac{b}{a}$$

$$C = \frac{2\pi\epsilon}{\ln(b/a)}$$

$$Z_0 = \sqrt{\frac{\frac{\mu}{2\pi}\ln b/a}{2\pi\epsilon/\ln(b/a)}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln(b/a)$$

$$= \frac{60}{\sqrt{\epsilon_r}}\ln\left(\frac{b}{a}\right) = 75$$

$$\frac{b}{a} = 6.52 \longrightarrow \boxed{b = 3.91\times 10^{-3}\,\text{m}}$$

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(9-18)

2.

$Z_L = 40 + j30\ \Omega$

$Z_0 = 50\ \Omega$

$f = 200\ \text{MHz}$

$l = 2\ \text{m}$

$$\beta l = \frac{2\pi f}{c} \cdot l = \frac{8\pi}{3} = 480^\circ$$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}$$

$$= 50 \frac{(40+j30) + j\,50(-1.732)}{50 + j(40+j30)(-1.732)}$$

$$\boxed{Z_{in} = 26.3 - j9.87\ \Omega}$$

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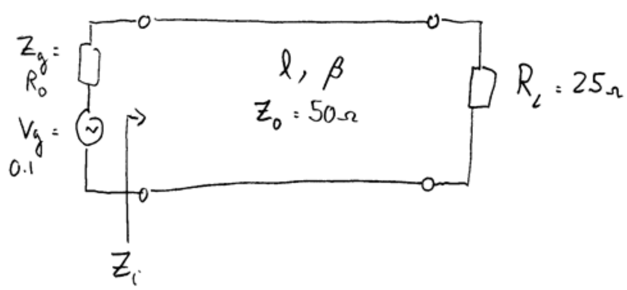
3. (9-30)



(a) $$Z_i = R_0 \left[\frac{R_L + R_0\, j \tan\beta l}{R_0 + R_L\, j \tan\beta l} \right]$$

$$= R_0 \left[\frac{1 + 2j \tan\beta l}{2 + j \tan\beta l} \right]$$

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$$\text{Then} \quad V_i = \frac{Z_i}{Z_i + Z_g} V_g = \frac{(1 + 2j\tan\beta\ell)}{3(1 + j\tan\beta\ell)} \times 0.1$$

$$I_i = \frac{V_i}{Z_i} = \frac{2 + j\tan\beta\ell}{3(1 + j\tan\beta\ell)} \times 2 \times 10^{-3}$$

$$\left.\begin{aligned} V_i &= V^+ + V^- \\ R_0 I_i &= V^+ - V^- \end{aligned}\right\} \Rightarrow V^+ = \frac{V_i + R_0 I_i}{2}$$

$$= \frac{1}{2}\, 0.1\, \frac{(3 + 3j\tan\beta\ell)}{3\cdot(1 + j\tan\beta\ell)} = 0.05\ \text{V}$$

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$$\text{At load:} \quad V_L^+ = V^+ e^{-j\beta\ell} = 0.05\, e^{-j\beta\ell}$$

$$\text{Then} \quad V_L^- = \Gamma_L V_L^+ = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) 0.05\, e^{-j\beta\ell}$$

$$= -\frac{0.05}{3} e^{-j\beta\ell}$$

$$V_L = V_L^+ + V_L^- = \frac{0.1}{3} e^{-j\beta\ell} = 0.0333\, e^{-j\beta\ell}$$

$$I_L = \frac{V_L}{Z_L} = \frac{0.1}{3 \times 25} e^{-j\beta\ell} = (1.33 \times 10^{-3})\, e^{-j\beta\ell}$$

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(b) $$\text{SWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$$

(c) $$\text{POWER} = \frac{1}{2}\,\text{Re}\, V_L I_L^* .$$

$$= \frac{1}{2}\,\text{Re}\,\frac{0.1}{3} \times \frac{0.1}{3 \times 25} = 0.0222\ \text{mW}$$

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If $R_L = 50$ (matched load):

$$V_L = V_i = \frac{R_0}{R_0+R_0} V_g = \frac{V_g}{2} = \frac{0.1}{2}\ \text{V}$$

$$\text{POWER} = \frac{1}{2}\,\text{Re}\,\frac{V_L V_L^*}{R_0} = \frac{1}{2}\,\frac{1}{50}\left|\frac{0.1}{2}\right|^2$$

$$= 0.025\ \text{mW}$$

i.e. more power to load when load is matched

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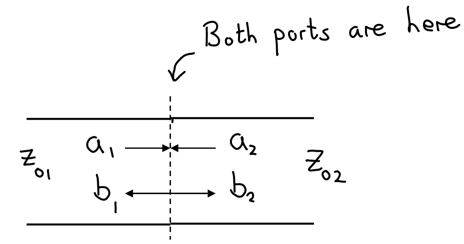
4. (a)



Step 1: Set $a_2 = 0$, $a_1 = 1$. Then

$$V_1 = \sqrt{Z_{01}}\,(1 + S_{11}) \qquad V_2 = \sqrt{Z_{02}}\,S_{21}$$

$$I_1 = (1 - S_{11})/\sqrt{Z_{01}} \qquad I_2 = -S_{21}/\sqrt{Z_{02}}$$

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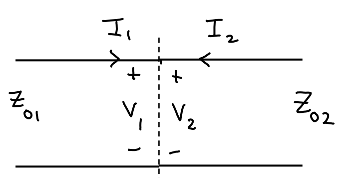
Step 2:



$$V_1 = V_2$$

$$I_1 = -I_2$$

Step 3: Substitute:

$$\sqrt{Z_{01}}\,(1 + S_{11}) = \sqrt{Z_{02}}\,S_{21}$$

$$(1 - S_{11})/\sqrt{Z_{01}} = +\ S_{21}/\sqrt{Z_{02}}$$

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Step 4: Solve

$$\sqrt{Z_{01}}\,(1+S_{11}) = \sqrt{Z_{02}}\,S_{21}$$
$$(1-S_{11})/\sqrt{Z_{01}} = +\,S_{21}/\sqrt{Z_{02}}$$

$$1+S_{11} = r\,S_{21} \qquad \text{where } r = \sqrt{\frac{Z_{02}}{Z_{01}}}$$
$$1-S_{11} = \frac{1}{r}\,S_{21}$$

$$\Rightarrow \quad \left(r+\frac{1}{r}\right)S_{21} = 2 \quad \Rightarrow \quad S_{21} = \frac{2r}{1+r^2} = \frac{2\sqrt{Z_{01}Z_{02}}}{Z_{02}+Z_{01}}$$

$$S_{11} = \frac{2r^2}{1+r^2} - 1 = \frac{r^2-1}{r^2+1} = \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}$$

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Step 5: $S_{12} = S_{21}$ (passive)

$$= \frac{2\sqrt{Z_{01}Z_{02}}}{Z_{02}+Z_{01}}$$

$S_{22} = S_{11}$ (symmetric)

$$= \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}$$

So

$$[S] = \begin{bmatrix} \dfrac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}} & \dfrac{2\sqrt{Z_{01}Z_{02}}}{Z_{02}+Z_{01}} \\[2ex] \dfrac{2\sqrt{Z_{01}Z_{02}}}{Z_{02}+Z_{01}} & \dfrac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}} \end{bmatrix}$$

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$$[S] = \begin{bmatrix} \dfrac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} & \dfrac{2\sqrt{Z_{01} Z_{02}}}{Z_{02} + Z_{01}} \\ \dfrac{2\sqrt{Z_{01} Z_{02}}}{Z_{02} + Z_{01}} & \dfrac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \end{bmatrix}$$

Check the case $Z_{01} = Z_{02}$:

$$[S] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$. Is this right?

With $Z_{01} = 100$, $Z_{02} = 50$:

$$[S] = \begin{bmatrix} -1/3 & 2\sqrt{2}/3 \\ 2\sqrt{2}/3 & -1/3 \end{bmatrix}$$

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4 (b) For a lossless device:

$$|S_{11}|^2 + |S_{21}|^2 = 1$$
$$|S_{12}|^2 + |S_{22}|^2 = 1$$

Here:

$$|S_{11}|^2 + |S_{21}|^2 = \frac{1}{9} + \frac{8}{9} = 1$$

$$|S_{12}|^2 + |S_{22}|^2 = \frac{8}{9} + \frac{1}{9} = 1$$

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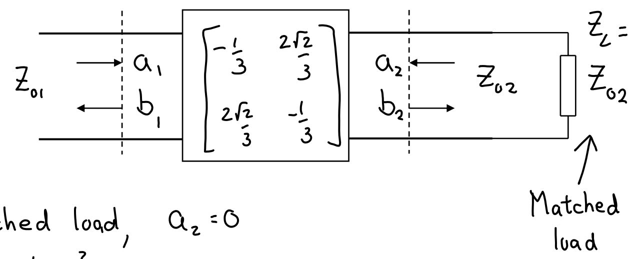
4 (c)



With matched load, $a_2 = 0$

$P_{inc} = \frac{1}{2} |a_1|^2$

$P_{absorbed} = \frac{1}{2} |b_2|^2$

$b_2 = S_{21} a_1 + S_{22} a_2 = S_{21} a_1 = \frac{2\sqrt{2}}{3} a_1$

So $\frac{P_{absorbed}}{P_{inc}} = \frac{|b_2|^2}{|a_1|^2} = \frac{8}{9}$ //

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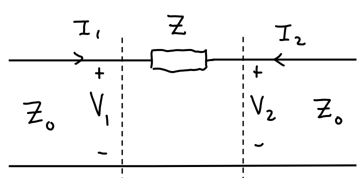
5.



Step 1: Set $a_2 = 0$, $a_1 = 1$. Then

$V_1 = \sqrt{Z_0}\,(1 + S_{11})$ $\qquad V_2 = \sqrt{Z_0}\, S_{21}$

$I_1 = (1 - S_{11}) / \sqrt{Z_0}$ $\qquad I_2 = -S_{21} / \sqrt{Z_0}$

Step 2: $I_1 = -I_2$

$V_1 = V_2 + I_1 Z$

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Step 3: Substitute

$I_1 = -I_2 \quad \Rightarrow \quad (1-S_{11})\sqrt{Z_0} = +S_{21}\sqrt{Z_0}$

$V_1 = V_2 + I_1 Z \quad \Rightarrow \quad \sqrt{Z_0}\,(1+S_{11}) = \sqrt{Z_0}\,S_{21} + (1-S_{11})\dfrac{Z}{\sqrt{Z_0}}$

Step 4: Solve

$1 - S_{11} = S_{21} \qquad r = Z/Z_0$

$1 + S_{11} = S_{21} + r(1-S_{11})$

$\Rightarrow \quad S_{11} = \dfrac{r}{2}(1-S_{11}) \;\Rightarrow\; \left(1+\dfrac{r}{2}\right)S_{11} = \dfrac{r}{2} \;\Rightarrow\; S_{11} = \dfrac{r}{2+r} = \dfrac{Z}{Z+2Z_0}$

$S_{21} = 1 - S_{11} = \dfrac{2Z_0}{Z+2Z_0}$

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Step 5 $\quad S_{12} = S_{21}$ (passive)

$S_{22} = S_{11}$ (symmetry)

$$[S] = \begin{bmatrix} \dfrac{Z}{Z+2Z_0} & \dfrac{2Z_0}{Z+2Z_0} \\[2ex] \dfrac{2Z_0}{Z+2Z_0} & \dfrac{Z}{Z+2Z_0} \end{bmatrix}$$

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