Problem Set 8

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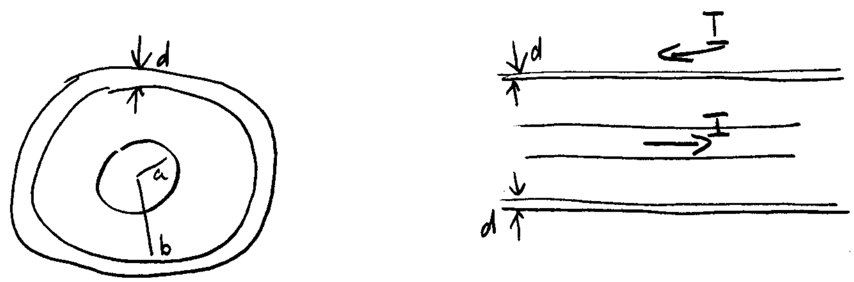
6-36

Find inductance per unit length.



3 regions: (1) inside inner conductor. $0 \le r \le a$

(2) Between inner & outer conductors $a \le r \le b$

(3) inside outer conductor $b \le r \le (b+d)$

(1) $$\bar{B}_1 = \bar{a}_\phi B_{\phi 1} = \bar{a}_\phi \frac{\mu r I}{2\pi a^2} \qquad 0 \le r \le a$$

(2) $$\bar{B}_2 = \bar{a}_\phi B_{\phi 2} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r} \qquad a \le r \le b$$

(3) $$\bar{B}_3 = \bar{a}_\phi B_{\phi 3} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}\left[1 - \frac{\pi(r^2-b^2)}{\pi(b+d)^2 - \pi b^2}\right]$$

$$= \bar{a}_\phi \frac{\mu_0 I}{2\pi r}\left[\frac{(b+d)^2 - r^2}{(b+d)^2 - b^2}\right] \qquad b \le r \le b+d$$

all from $\oint_C \bar{B} \cdot d\bar{l} = \mu I$

Magnetic energy per unit length stored in 3 regions is,

60

$$W_m = \frac{1}{2}\int_{V'} \frac{B^2}{\mu}\, dv' \quad (J)$$

$$W_{m1}' = \frac{1}{2\mu_0}\int_0^a B_{\phi 1}^2\, 2\pi r\, dr = \frac{\mu_0 I^2}{4\pi a^2}\int_0^a r^3\, dr = \frac{\mu_0 I^2}{16\pi} \quad \left(J/m\right)$$

$$W_{m2}' = \frac{1}{2\mu_0}\int_a^b B_{\phi 2}^2\, 2\pi r\, dr = \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{1}{r}\, dr = \frac{\mu_0 I^2}{4\pi}\ln b/a \quad \left(J/m\right)$$

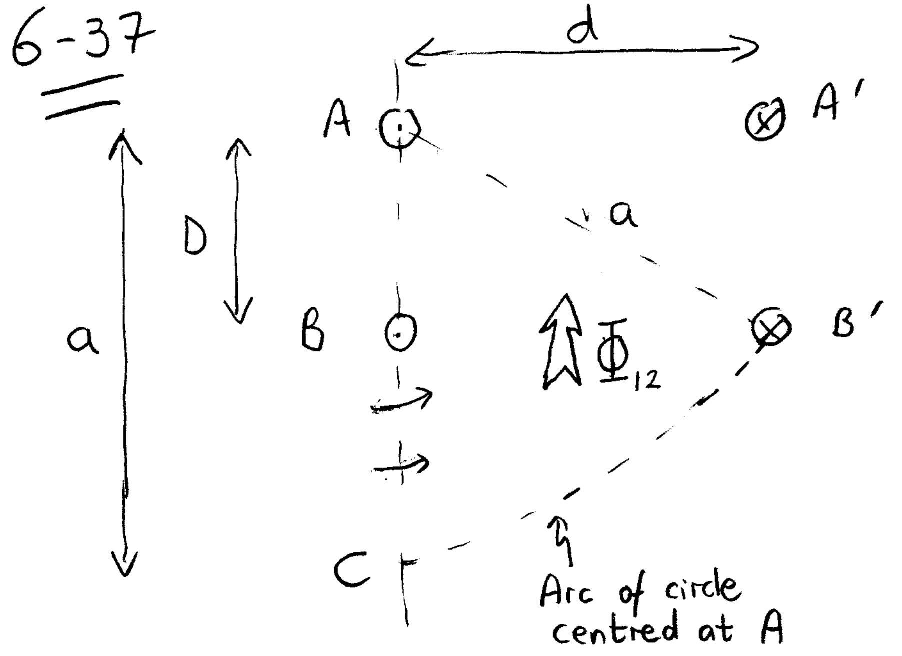
$$W_{m3}' = \frac{1}{2\mu_0}\int_b^{b+d} B_{\phi 3}^2\, 2\pi r\, dr = \frac{\mu_0 I^2}{4\pi}\left\{\frac{(b+d)^4}{[(b+d)^2-b^2]^2}\ln\left(1+\frac{d}{b}\right) + \frac{b^2-3(b+d)^2}{4[(b+d)^2-b^2]}\right\} \quad J/m$$

$$L = \frac{2W_m}{I^2} \quad \text{so} \quad L' = \frac{2W_m'}{I^2} \quad H/m$$

$$L' = \frac{2}{I^2}\left(W_{m1}' + W_{m2}' + W_{m3}'\right)$$

$$\boxed{L' = \frac{\mu_0}{2\pi}\left\{\frac{1}{4} + \ln b/a + \frac{(b+d)^4}{[(b+d)^2-b^2]}\ln\left(1+\frac{d}{b}\right) - \frac{b^2-3(b+d)^2}{4[(b+d)^2-b^2]}\right\} \quad (H/m)}$$

6-37



AA' = Loop 1

BB' = Loop 2

$a = AB' = \sqrt{D^2 + d^2}$

$\bar{B}$ from infinitely long line carrying a current I: $\bar{B} = \bar{a}_\varphi \dfrac{\mu_0 I}{2\pi r}$

Let current I_1 flow in loop AA'.
Need to find flux Φ_{12} linking BB'
Start with flux Φ_A due to just current in A.

$$\Phi_A = \underbrace{\cancel{\int_{\text{Arc } B'C} \bar{B} \cdot d\bar{s}}}_{0} + \int_{\text{Line } CB} \bar{B} \cdot d\bar{s}$$

0 because $\bar{B}$ is tangent to arc

NOTE Choose unusual surface ($BC + CB'$) instead of flat surface (BB') to make calculation easier

$$= \int_{r=D}^{a} \frac{\mu_0 I_1}{2\pi r}\, dr = \frac{\mu_0 I_1}{2\pi} \ln \frac{a}{D}$$

Similarly, flux $\Phi_{A'}$ due to current in A' is

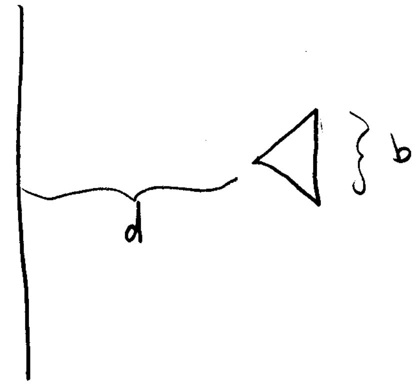
$$\Phi_{A'} = \int_{r=D}^{a} \frac{\mu_0 I_1}{2\pi r}\, dr = \frac{\mu_0 I_1}{2\pi} \ln \frac{a}{D}$$

$$\Phi_{12} = \Phi_A + \Phi_{A'} = \frac{\mu_0 I_1}{2\pi} \ln \frac{a^2}{D^2}$$

$$= \frac{\mu_0 I_1}{2\pi} \ln \frac{D^2+d^2}{D^2}$$

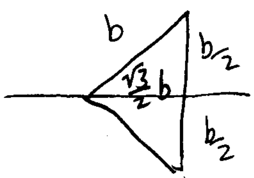
So mutual inductance $= \frac{\Lambda_{12}}{I_1} = \frac{\Phi_{12}}{I_1} = \frac{\mu_0}{2\pi} \ln\left(1 + \frac{d^2}{D^2}\right)$

6-38

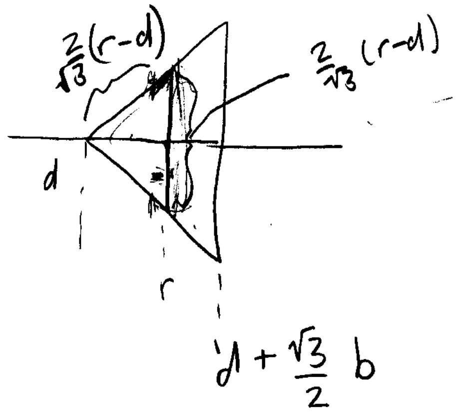


I for long straight wire $\rightarrow$ $\bar{B} = \bar{a}_\phi \dfrac{\mu_0 I}{2\pi r}$

$$\Lambda_{12} = \int_{S_2} \bar{B}_1 \cdot d\bar{s}_2 = \int B_\phi \overbrace{\frac{2}{\sqrt{3}}(r-d)\,dr}^{ds}$$



and



$$\Lambda_{12} = \frac{\mu_0 I}{\cancel{2}\pi} \int_d^{d+\frac{\sqrt{3}}{2}b} \frac{\cancel{2}}{\sqrt{3}} \frac{(r-d)}{r}\,dr = \frac{\mu_0 I}{\pi\sqrt{3}} \int_d^{d+\frac{\sqrt{3}}{2}b} \left(\frac{r-d}{r}\right) dr$$

$$= \frac{\mu_0 I}{\pi\sqrt{3}} \left[\frac{\sqrt{3}}{2} b - d \ln\left(1 + \frac{\sqrt{3}b}{2d}\right)\right]$$

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\mu_0}{\pi}\left[\frac{b}{2} - \frac{d}{\sqrt{3}} \ln\left(1 + \frac{\sqrt{3}d}{2b}\right)\right]$$

6-41

(64)

$$W_2 = \tfrac{1}{2} L_1 I_1^2 + M I_1 I_2 + \tfrac{1}{2} L_2 I_2^2$$

a) $$W_2 = \frac{I_2^2}{2}\left[L_1\left(\frac{I_1}{I_2}\right)^2 + 2M\left(\frac{I_1}{I_2}\right) + L_2\right]$$

$$= \frac{I_2^2}{2}\left(L_1 x^2 + 2Mx + L_2\right), \quad x = \frac{I_1}{I_2}$$

$$\frac{dW_2}{dx} = \frac{I_2^2}{2}(2L_1 x + 2M) = 0 \longrightarrow x = -\frac{M}{L_1}$$

$$\frac{d^2 W_2}{dx^2} = I_2^2 L_1 > 0$$ to be a minimum and $L_1 > 0$ and $I_2^2 > 0$ so yes

$$x = \frac{I_1}{I_2} = -\frac{M}{L_1}$$ for minimum W_2

b) Show $M \leq \sqrt{L_1 L_2}$

$$(W_2)_{min} = \frac{I_2^2}{2}\left(-\frac{M^2}{L_1} + L_2\right) \geq 0$$

so $$-\frac{M^2}{L_1} + L_2 \geq 0$$

$$-M^2 + L_1 L_2 \geq 0$$

$$M^2 \leq L_1 L_2$$

$$\underline{M \leq \sqrt{L_1 L_2}}$$