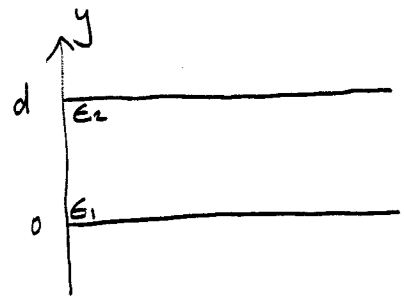
PROBLEM SET #5

(37)

3-30, 3-32, 3-34, 3-40

3-30



Find C.

ϵ varies linearly from $y=0$ to $y=d$,

so find expression for ϵ...

$$\frac{\epsilon_2 - \epsilon_1}{y_2 - y_1} = k \text{ (constant slope)}$$

$$\frac{\epsilon_2 - \epsilon_1}{d - 0} = k \rightarrow k = \frac{\epsilon_2 - \epsilon_1}{d}$$

For general point, $\dfrac{\epsilon - \epsilon_1}{y - \cancel{y_1}_0} = k$

$$\frac{\epsilon - \epsilon_1}{y} = \frac{\epsilon_2 - \epsilon_1}{d}$$

$$\underline{\epsilon = \frac{\epsilon_2 - \epsilon_1}{d} y + \epsilon_1}$$

Assume Q on plate at $y=d$

so $$\bar{E} = \underbrace{-\bar{a}_y \frac{\rho_s}{\epsilon}}_{\text{from Gauss Law}} = -\bar{a}_y \frac{\rho_s}{\frac{\epsilon_2 - \epsilon_1}{d} y + \epsilon_1}$$

Now, $V = -\int_{y=0}^{y=d} \bar{E} \cdot d\bar{l} = \cancel{-}\int_{y=0}^{y=d} \cancel{-}\cancel{\bar{a}_y} \frac{\rho_s}{\frac{\epsilon_2 - \epsilon_1}{d} y + \epsilon_1} \cdot \cancel{\bar{a}_y}\, dy$

*note
$\rho_s = \frac{Q}{S}$

$$V = \frac{d}{\epsilon_2 - \epsilon_1} \int_0^d \frac{\rho_s}{\frac{\epsilon_2 - \epsilon_1}{d} y + \epsilon_1} \, d\left(\frac{\epsilon_2 - \epsilon_1}{d} y \right) = \frac{d}{\epsilon_2 - \epsilon_1} \frac{Q}{S} \left[\ln \left(\frac{\epsilon_2 - \epsilon_1}{d} y + \epsilon_1 \right) \right]_0^d$$

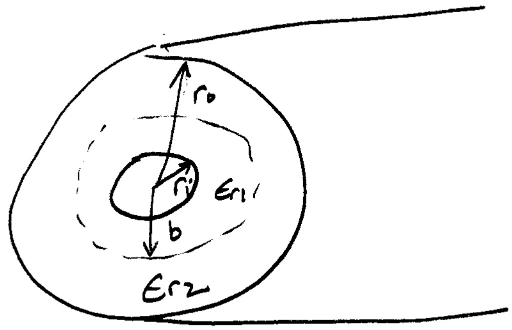
$$V = \frac{d}{(\epsilon_2 - \epsilon_1)} \frac{Q}{S} \left[\ln\left(\frac{(\epsilon_2 - \cancel{\epsilon_1})\cancel{d}}{\cancel{d}} + \cancel{\epsilon_1} \right) - \ln \epsilon_1 \right]$$

$$V = \frac{Q d \ln(\epsilon_2 / \epsilon_1)}{S(\epsilon_2 - \epsilon_1)}$$

$$\boxed{C = \frac{Q}{V} = \frac{S(\epsilon_2 - \epsilon_1)}{d \ln(\epsilon_2 / \epsilon_1)} \text{ (F)}}$$

3-32

Find C/unit length



For coax, $\bar{D} = \bar{a}_r \dfrac{\rho_\ell}{2\pi r}$ (can be derived w/ Gauss's Law → did this a few assigments ago!)

$\bar{D} = \epsilon \bar{E}$, so $\bar{E}_1 = \bar{a}_r \dfrac{\rho_\ell}{2\pi \epsilon_0 \epsilon_{r1} r}$, $r_i < r < b$

$\bar{E}_2 = \bar{a}_r \dfrac{\rho_\ell}{2\pi \epsilon_0 \epsilon_{r2} r}$, $b < r < r_o$

$$V = -\int_{r_o}^{r_i} \bar{E} \cdot \underbrace{d\bar{\ell}}_{d\bar{r}} = -\left[\int_{r_o}^{b} \cancel{\bar{a}_r} \frac{\rho_\ell}{2\pi \epsilon_2 r} \cdot \cancel{\bar{a}_r}\, dr + \int_{b}^{r_i} \cancel{\bar{a}_r} \frac{\rho_\ell}{2\pi \epsilon_1 r} \cdot \cancel{\bar{a}_r}\, dr \right]$$

$$V = -\frac{\rho_\ell}{2\pi \epsilon_0} \left(\frac{1}{\epsilon_{r2}} \left[\ln r \right]_{r_o}^{b} + \frac{1}{\epsilon_{r1}} \left[\ln r \right]_{b}^{r_i} \right)$$

$$V = -\frac{\rho_\ell}{2\pi \epsilon_0} \left(\frac{1}{\epsilon_{r2}} \left[\ln b - \ln r_o \right] + \frac{1}{\epsilon_{r1}} \left[\ln r_i - \ln b \right] \right)$$

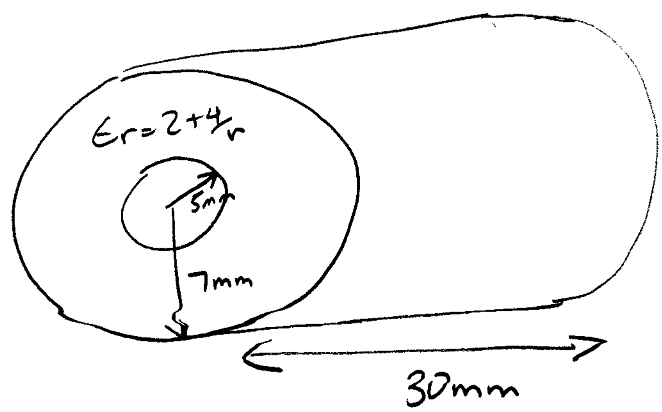
$$V = \frac{\rho_\ell}{2\pi \epsilon_0} \left(\frac{1}{\epsilon_{r2}} (\ln r_o - \ln b) + \frac{1}{\epsilon_{r1}} (\ln b - \ln r_i) \right)$$

$$V = \frac{\rho_\ell}{2\pi \epsilon_0} \left[\frac{1}{\epsilon_{r2}} \ln (r_o/b) + \frac{1}{\epsilon_{r1}} \ln (b/r_i) \right]$$

$$C = \frac{Q}{V}, \quad C' = \frac{\overset{\text{per unit length}}{\rho_l}}{V} \qquad \text{why? } Q = \int \rho_l \, dl = \rho_l L$$

$$\boxed{C' = \frac{2\pi\epsilon_0}{\left[\frac{1}{\epsilon_{r1}} \ln\left(\frac{b}{r_i}\right) + \frac{1}{\epsilon_{r2}} \ln\left(\frac{r_o}{b}\right)\right]} \quad (F/m)}$$

3-34



Find C.

$$\bar{E} = \bar{a}_r \frac{\rho_l}{2\pi\epsilon r} = \bar{a}_r \frac{\rho_l}{2\pi\epsilon_0 (2 + \frac{4}{r}) r}$$

$$\bar{E} = \bar{a}_r \frac{\rho_l}{4\pi\epsilon_0 (r+2)}$$

$$V = -\int_{r_o}^{r_i} \bar{E} \cdot d\bar{r} = -\int_{r_o}^{r_i} \cancel{\bar{a}_r} \frac{\rho_l}{4\pi\epsilon_0 (r+2)} \cdot \cancel{\bar{a}_r} \, dr$$

$$V = -\frac{\rho_l}{4\pi\epsilon_0} \int \frac{1}{(r+2)} d(r+2) = -\frac{\rho_l}{4\pi\epsilon_0} \Big[\ln(r+2)\Big]_{r_o}^{r_i}$$

$$V = \frac{\rho_\ell}{4\pi\epsilon}\left[\ln(r_o+2) - \ln(r_i+2)\right]$$

$$V = \frac{\rho_\ell}{4\pi\epsilon_0}\ln\left(\frac{r_o+2}{r_i+2}\right) = \frac{\rho_\ell}{4\pi\epsilon_0}\ln\left(\frac{7+2}{5+2}\right)$$

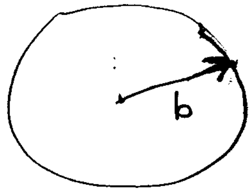
$$\underline{V = \frac{\rho_\ell}{4\pi\epsilon_0}\ln\left(\frac{9}{7}\right)}$$

$$C = \frac{Q}{V}, \quad Q = \rho_\ell L$$

$$C = \frac{\rho_\ell L}{V} = \frac{4\pi\epsilon_0 \overset{30mm}{L}}{\ln\left(\frac{9}{7}\right)} = 1500\,\epsilon_0$$

$$\boxed{C = 13.26\ nF} = 13.26\times10^{-9}\ F$$

3-40



$$\bar{D} = \bar{a}_R \frac{R}{3}\rho , \quad R < b$$

$$\bar{D} = \bar{a}_R \frac{b^3\rho}{3R^2} , \quad R > b \qquad \bar{E} = \frac{\bar{D}}{\epsilon_0}$$

a) $$W_{inside} = \frac{1}{2}\int_V \bar{D}\cdot\bar{E}\, dv = \frac{1}{2}\int_0^{\overset{\text{inside the sphere}}{b}} \frac{1}{\epsilon_0}\left(\frac{R}{3}\rho\right)^2 \overbrace{4\pi R^2}^{\text{Surface area}} dR$$

$$= \frac{1}{\cancel{2}\epsilon_0}\int \frac{R^2\rho^2}{9}\cdot \overset{2}{\cancel{4}}\pi \cdot R^2\, dR$$

$$= \frac{2\pi\rho^2}{9\epsilon_0}\int_0^b R^4\, dR = \frac{2\pi\rho^2}{9\epsilon_0}\left[\frac{R^5}{5}\right]_0^b$$

$$\boxed{W_{inside} = \frac{2\pi b^5 \rho^2}{45\epsilon_0}}$$

b) $$W_{outside} = \frac{1}{2}\int_b^{\overset{\text{outside the sphere}}{\infty}} \frac{1}{\epsilon_0}\left(\frac{b^3\rho}{3R^2}\right)^2 4\pi R^2\, dR = \frac{1}{\cancel{2}\epsilon_0}\int \frac{b^6\rho^2}{9R^{\cancel{4}\,2}} \overset{2}{\cancel{4}}\pi \cancel{R^2}\, dR$$

$$= \frac{2b^6\rho^2\pi}{9\epsilon_0}\int_b^\infty \frac{1}{R^2}\, dR = \frac{2\pi b^6\rho^2}{9\epsilon_0}\left[-\frac{1}{R}\right]_b^\infty$$

$$= \frac{2\pi b^6\rho^2}{9\epsilon_0}\left[\frac{1}{b} - \overset{0}{\cancel{\frac{1}{\infty}}}\right] = \frac{2\pi b^5\rho^2}{9\epsilon_0}$$

$$\boxed{W_{outside} = \frac{2\pi b^5 \rho^2}{9\epsilon_0}}$$

To check w/ example 3-22,

Total $W = W_{inside} + W_{outside}$

$$W = \frac{2\pi b^5 \rho^2}{45\epsilon_0} + \frac{2\pi b^5 \rho^2}{9\epsilon_0}$$

$$W = \frac{\cancel{12}^{\,4}\,\pi b^5 \rho^2}{\cancel{45}_{\,15}\,\epsilon_0} = \boxed{\frac{4\pi b^5 \rho^2}{15\epsilon_0}}$$

Same as ex. 3-22