Problem Set 03

Solutions

ProblemSet_03: 1

3-13

$q = -2\,\mu C$

$\rightarrow$ find work done from $P_1(2,1,-1)$ to $P_2(8,2,-1)$

$\bar{E} = \bar{a}_x y + \bar{a}_y x$

$$\frac{W}{q} = -\int_{P1}^{P2} \bar{E}\cdot d\bar{l} \quad (J/_C)$$

$$\frac{W}{q} = -\int_{P1}^{P2} (\bar{a}_x y + \bar{a}_y x)\cdot \underbrace{(\bar{a}_x dx + \bar{a}_y dy + \bar{a}_z dz)}_{d\bar{l}}$$

$$\frac{W}{-q} = \int_{P1}^{P2} (y\,dx + x\,dy)$$

ProblemSet_03: 2

a) $x = 2y^2 \quad \frac{dx}{dy} = 4y \longrightarrow dx = 4y\,dy$

$$\frac{W}{-q} = \int_{P1}^{P2} y(4y\,dy) + (2y^2)dy$$

$$\frac{W}{-q} = \int_{P1}^{P2} 6y^2 dy = \int_{y_1=1}^{y_2=2} 6y^2 dy = \cancel{6}\left[\frac{y^3}{\cancel{3}}\right]_1^2 = 2[8-1]$$

$$\Longrightarrow W = -14q = -14(-2\mu\cancel{C})\frac{J}{\cancel{C}} = \underline{\underline{28\mu J}}$$

ProblemSet_03: 3

b) along straight line joining P_1 & P_2

What is eqn of that line?

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{8-2} = \frac{1}{6}$$

eqn of line that connects P1 & P2

Now eqn $\rightarrow \frac{y-1}{x-2} = \frac{1}{6} \longrightarrow x = 6y - 4$

$\frac{dx}{dy} = 6 \quad dx = 6\,dy$

$$\frac{W}{-q} = \int_{P_1}^{P_2} y6dy + (6y-4)dy = \int_{y_1=1}^{y_2=2} (12y-4)dy$$

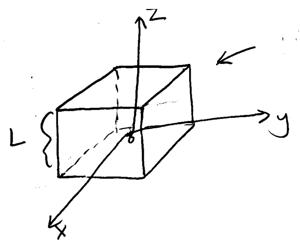
$$\frac{W}{-q} = \left[\frac{\overset{6}{\cancel{12}}y^2}{\cancel{2}} - 4y\right]_1^2 = 14$$

$$W = -14q\,\frac{J}{C} = -14(-2\mu\cancel{C})\,\frac{J}{\cancel{C}} = 28\mu J$$

$$\underline{\underline{W = 28\mu J}}$$

ProblemSet_03: 4

3-22



inaccurate sketch of cube w/ center at origin

$$P = P_0 (\bar{a}_x x + \bar{a}_y y + \bar{a}_z z)$$

a) find surface and volume bound-charge densities

$$\rho_{ps} = \bar{P} \cdot \bar{a}_n$$

$$\rho_p = -\nabla \cdot \bar{P}$$

ProblemSet_03: 5

$$\rho_{ps} = P_0 (\bar{a}_x x + \bar{a}_y y + \bar{a}_z z) \cdot \bar{a}_n$$

$\bar{a}_{n1} = \bar{a}_x$, $\bar{a}_{n2} = -\bar{a}_x$, $\bar{a}_{n3} = \bar{a}_y$, etc ... for all 6 faces.

so ρ_{ps} will be same for all faces because cube is centered at origin

Take $\bar{a}_n = \bar{a}_x$

$$\rho_{ps} = P_0 (\bar{a}_x x + \bar{a}_y y + \bar{a}_z z) \cdot \bar{a}_x = P_0 x$$

but $x = \frac{L}{2}$ (centered at origin)

so $\boxed{\rho_{ps} = P_0 \frac{L}{2}}$ for all 6 faces

ProblemSet_03: 6

To check, calculate ρ_{ps} for back face at $z = -\frac{L}{2}$

$$\rho_{ps} = P_o(\bar{a}_x x + \bar{a}_y y + \bar{a}_z z) \cdot (-\bar{a}_z) = -P_o z$$

$$z = -\frac{L}{2}$$

so $\rho_{ps} = P_o \frac{L}{2}$ → same as it would be if calculated for any other side

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$$\rho_p = -\nabla \cdot P$$

$$\rho_p = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right)$$

$$\rho_p = -\left(\frac{\partial\, x P_o}{\partial x} + \frac{\partial\, y P_o}{\partial y} + \frac{\partial\, z P_o}{\partial z} \right)$$

$$\boxed{\rho_p = -3 P_o}$$

ProblemSet_03: 7

b) Show that the total bound charge is zero.

$$Q_{\text{total bound}} = Q_s + Q_v$$

$$Q_s = \oint_S \rho_{ps}\, ds = \oint_S P_o \frac{L}{2}\, ds = P_o \frac{L}{2} \underbrace{\oint_S ds}_{\text{area of cube}}$$

$$Q = P_o \frac{L}{2} (\underbrace{6}_{\#\text{sides of cube}} \underbrace{L^2}_{\text{area of 1 side}})$$

$$Q_s = 3 P_o L^3$$

ProblemSet_03: 8

$$Q_V = \int_V \rho_e \, dv = \int_V -3P_0 \, dv = -3P_0 \int dv$$

volume

$$Q_V = -3P_0 \left(L^3\right)$$

volume of cube

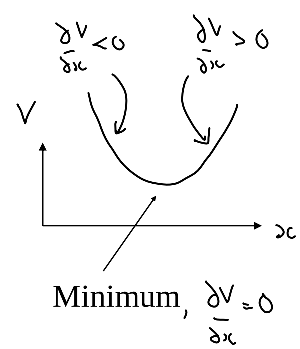
so $Q_s + Q_V = 3P_0L^3 + (-3P_0L^3) = \underline{\underline{0}}$

shown!

ProblemSet_03: 9

4-3

At a minimum of V, the slope dV/dx is increasing, and so:



$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial^2 V}{\partial x^2} > 0$$

ProblemSet_03: 10

For the same reason: $\frac{\partial^2 V}{\partial y^2} > 0 \quad , \quad \frac{\partial^2 V}{\partial z^2} > 0$

Then at a minimum we must have:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} > 0$$

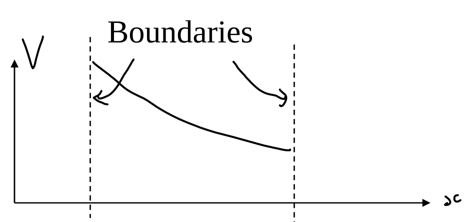
But Laplace's equation is: $\nabla^2 V = 0$

So V cannot have a minimum within a region where it is governed by the Laplace equation. (It can be a minimum on the *boundary* of such a region, however).

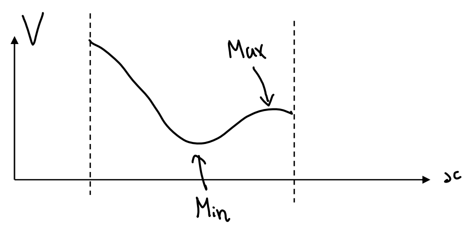
In the same way, it can be shown that a *maximum* is not allowed within the region.

ProblemSet_03: 11

In other words, Laplacian solutions look like this:



<u>not</u> like this:



(Compare this with solutions to the wave equation, which you will see later)

ProblemSet_03: 12

4-4 Verify that V_1 & V_2 are solutions of Laplace's equation.

In other words, show $\nabla^2 V_1 = 0$
& $\nabla^2 V_2 = 0$

$\underline{\underline{V_1}}$ $V_1 = C_1/R \rightarrow$ spherical coords

V_1 is only function of R

$$\nabla^2 V_1 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V_1}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \cancel{\frac{\partial V_1}{\partial \theta}}\right)^{0} + \frac{1}{R^2\sin^2\theta}\left(\cancel{\frac{\partial^2 V_1}{\partial \phi^2}}\right)^{0}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\frac{C_1}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(\cancel{R^2}\left(-\frac{C_1}{\cancel{R^2}}\right)\right)$$

$$= -\frac{1}{R^2}\underbrace{\cancel{\frac{\partial}{\partial R}(C_1)}}_{0} = 0 \quad \text{so } \nabla^2 V_1 = 0 \checkmark$$

ProblemSet_03: 13

$\underline{\underline{V_2}}$

$$V_2 = \frac{C_2 z}{(x^2+y^2+z^2)^{3/2}} = C_2 z (x^2+y^2+z^2)^{-3/2}$$

$$\nabla^2 V_2 = \frac{\partial^2}{\partial x^2} V_2 + \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial^2 V_2}{\partial z^2}$$

$$\frac{\partial V}{\partial x} = -3C_2 xz(x^2+y^2+z^2)^{-\frac{3}{2}-1}$$

$$\frac{\partial V}{\partial y} = -3C_2 yz(x^2+y^2+z^2)^{-\frac{3}{2}-1}$$

$$\frac{\partial V}{\partial z} = -3C_2 zz(x^2+y^2+z^2)^{-\frac{3}{2}-1} + C_2(x^2+y^2+z^2)^{-3/2}$$

ProblemSet_03: 14

$$\frac{\partial^2 V}{\partial x^2} = -3C_2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-1} + 3\cdot 5\cdot C_2 x^2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-2}$$

$$\frac{\partial^2 V}{\partial y^2} = -3C_2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-1} + 3\cdot 5\cdot C_2 y^2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-2}$$

$$\frac{\partial^2 V}{\partial z^2} = -6C_2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-1} + 3\cdot 5\cdot C_2 z^2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-2} - 3C_2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-1}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -15C_2 z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-1} + 15C_2 z\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-\frac{3}{2}-2}$$

$$= \underline{\underline{0}} \quad \checkmark$$

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