Problem Set 2

(10)

3-10, 11, 12

3-10

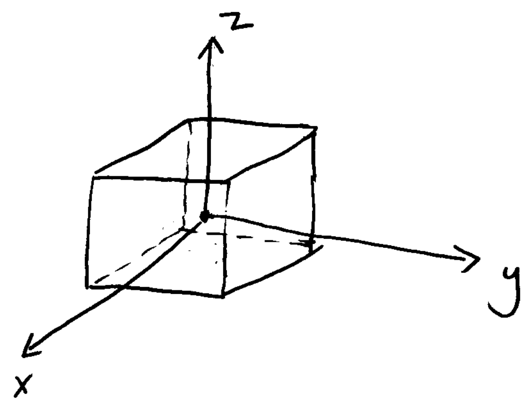
$$\bar{E} = \bar{a}_x \, 100x \quad (V/m)$$

Find the total electric charge ----

Gauss's Law $\rightarrow$ $\oint_S \bar{E} \cdot d\bar{s} = \frac{Q}{\varepsilon_0}$ total charge

$$Q = \epsilon_0 \oint_S \bar{E} \cdot d\bar{s}$$

a)



$\bar{E} = \bar{a}_x \, 100x \; (V/m)$

$d\bar{s} = \bar{a}_n \, ds$

$\bar{a}_n$ = outward normal to surface enclosing cube

so $\bar{a}_{n_1} = \bar{a}_x$

$\bar{a}_{n_2} = -\bar{a}_x$

$\bar{a}_{n_3} = \bar{a}_y$

$\bar{a}_{n_6} = -\bar{a}_z$

but $Q = \epsilon_0 \oint_S \bar{E} \cdot \bar{a}_n \, ds$

since $\bar{E} = 100x \, \bar{a}_x$, and
$\bar{a}_x \cdot \bar{a}_y = 0$
$\bar{a}_x \cdot \bar{a}_z = 0$
$\bar{a}_x \cdot \bar{a}_x = 1$

only sides at $x = \pm 50$mm contribute

→ at $x = +50$mm → $\bar{E}_1 = 100x \, \bar{a}_x = 100 \cdot 0.05\text{m} \, a_x$
$= 5 \bar{a}_x$

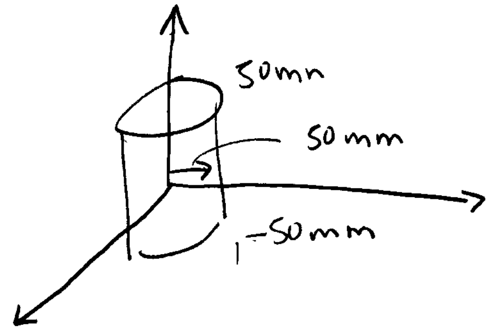
→ at $x = -50$ mm → $\bar{E}_2 = -5\bar{a}_x$

$$Q = \epsilon_0 \oint \bar{E} \cdot d\bar{s} = \epsilon_0 \iint_{S_1} 5 \bar{a}_x \cdot \bar{a}_x \, ds + \iint_{S_2} (-5\bar{a}_x) \cdot (-\bar{a}_x) \, ds$$

$$= \epsilon_0 \left[(0.1)^2 \cdot 5 + (0.1)^2 \cdot 5 \right] = 0.1 \epsilon_0$$

$$\underline{Q = 8.84 \times 10^{-13} \ (C)}$$

b)



$$\bar{E} = \bar{a}_x \, 100x \ \frac{V}{m}$$

Surface that the cylinder encloses is obviously easiest expressed in cylindrical coordinates.

Since we are going to have to perform $\bar{E} \cdot \bar{ds}$, we should express $\bar{E}$ in cylindrical coords.

$$\bar{E} = \alpha_1 \bar{a}_r + \alpha_2 \bar{a}_\phi + \cancel{\alpha_3 \bar{a}_z}$$ does not change w/ z

What are contributions to $\bar{a}_x$ from $\bar{a}_r$ & $\bar{a}_\phi$?

From $\bar{a}_r$, $\bar{a}_r \cdot 100x\bar{a}_x = 100x\cos\phi$

$\bar{a}_\phi$, $\bar{a}_\phi \cdot 100x\bar{a}_x = -100x\sin\phi$

So $$\bar{E} = 100x\cos\phi \, \bar{a}_r - 100x\sin\phi \, \bar{a}_\phi$$

$$\bar{a}_n = \bar{a}_r$$

so $\bar{E} \cdot \bar{a}_r = E_r$

$ds = r\,d\phi\,dz$

so $$Q = \epsilon_0 \oint_S \bar{E} \cdot \bar{a}_r \, ds = \epsilon_0 \iint_S 100x \cos\phi \, r\,d\phi\,dz$$

but $x = r\cos\phi$

(13)

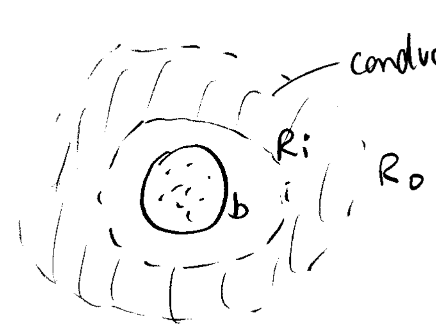
$$Q = \epsilon_0 \int_0^{2\pi}\int_{-50mm}^{50mm} 100\, r^2 \cos^2\phi \, d\phi \, dz$$

$$Q = 100\epsilon_0 r^2 \int_{-50mm}^{50mm} dz \int_0^{2\pi} \cos^2\phi \, d\phi$$

$$Q = 100\epsilon_0 (r^2)_{r=50mm} (z)_{z=100mm} \left[\tfrac{1}{2}\phi + \tfrac{1}{4}\sin 2\phi\right]_0^{2\pi}$$

$$Q = 0.025\pi\epsilon_0 = 6.94\times10^{-13}$$

3-11 $\qquad \rho = \rho_0 \left[1 - \frac{R^2}{b^2}\right]$ $\qquad$ 14



find $\bar{E}$

Because of spherical symmetry,

$$\bar{E} = \bar{a}_R E_R$$

To find $\bar{E}$, apply Gauss's Law

1) $\quad 0 \le R \le b \qquad \bar{E} = \bar{a}_r E_{R1} \qquad d\bar{s} = \bar{a}_R \, ds$

$$\oint \bar{E} \cdot d\bar{s} = \underbrace{E_{R1}}_{\text{const. on that surface}} \int_S ds = E_{R1} 4\pi R^2$$

$$Q = \int_V \rho \, dv = \rho_0 \int_V \left[1 - \frac{R^2}{b^2}\right] dv$$

because this is the source ρ

$$Q = \rho_0 \int_0^R \int_0^{2\pi} \int_0^{2\pi} \left[1 - \frac{R^2}{b^2}\right] dv = \rho_0 \int_0^R \left[1 - \frac{R^2}{b^2}\right] 4\pi R^2 dR$$

$$Q = 4\pi \rho_0 \left[\frac{R^3}{3} - \frac{R^5}{b^2 5}\right]_0^R = 4\pi \rho_0 \left(\frac{R^3}{3} - \frac{R^5}{5b^2}\right)$$

$$\oint \bar{E}\cdot d\bar{s} = \frac{Q}{\varepsilon_0}$$

$$E_{R1} 4\pi R^2 = \frac{Q}{\varepsilon_0}$$

$$\boxed{E_{R1} = \frac{\rho_0}{\epsilon_0} R\left(\frac{1}{3} - \frac{R^2}{5b^2}\right)}$$

2) $b \geq R \leq R_i$: $\bar{E} = E_{R2}\,\bar{a}_R$

Similarly

limits of the source of total charge Q → remember ρ exists only from $0 \leq R \leq b$ but $\bar{E}$ is everywhere.

$$4\pi R^2 E_{R2} = \frac{\rho_0}{\epsilon_0}\int_0^b \left(1 - \frac{R^2}{b^2}\right) 4\pi R^2 dR$$

$$\boxed{E_{R2} = \frac{2\rho_0 b^3}{15\epsilon_0 R^2}}$$

3) $R_i < R < R_o$

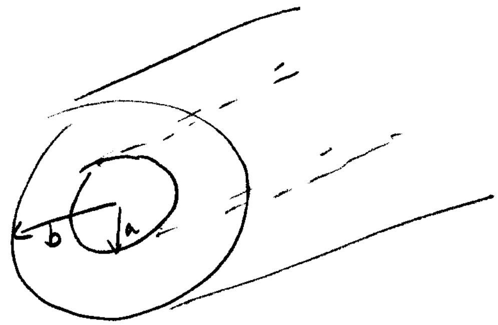
↳ $E_{R3} = 0$ → 0 field in Conductors!

4) $R > R_o$

Exactly same as region 2

$$\boxed{E_{R4} = \frac{2\rho_0 b^3}{15\varepsilon_0 R^2}}$$

3-12



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Cylindrical symmetry: $\bar{E} = \bar{a_r} E_r$

Apply Gauss's Law

a) E?

$$\oint E \cdot ds = E_r \int ds$$

i) $r < a \rightarrow E_r = 0$

ii) $a < r < b$

$$\oint \bar{E} \cdot \bar{ds} = \frac{Q}{\epsilon_0}$$

$\rightarrow$ due to surface charge

$$\Longrightarrow E_r \int_S ds = \frac{1}{\epsilon_0} \int_S \rho_{sa} \, ds$$

$$E_r (2\pi r \cancel{z}) = \frac{\rho_{sa}}{\epsilon_0} (\cancel{2\pi} a \cancel{z})$$

$$\boxed{E_r = \frac{\rho_{sa}\, a}{\epsilon_0\, r}}$$

iii) $r > b$, same as ii) except

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we have $\bar{E}_r$ due to surface charges

ρ_{sa} & ρ_{sb} so

$$\boxed{E_r = \frac{a\rho_{sa} + b\rho_{sb}}{\epsilon_0 r}}$$

b) for $r > b$, what should relation be between a & b to make $\bar{E} = 0$?

$$\bar{E} = 0 \rightarrow E_r = 0 = \frac{a\rho_{sa} + b\rho_{sb}}{\cancel{\epsilon_0 r}} = 0$$

$$a\rho_{sa} = -b\rho_{sb}$$

$$\boxed{\frac{b}{a} = -\frac{\rho_{sa}}{\rho_{sb}}}$$